

An Introduction to Kan Extensions: Playing with Posets

hora-algebra

English translation by Astra

This introduction assumes no knowledge of category theory. We use Kan extensions between posets to explain the idea of a Kan extension in elementary terms. The examples come from a variety of fields and range widely in difficulty: the aim is to understand the idea by working through toy examples ourselves. We have a broad readership in mind, from high school students who know little or no modern mathematics, let alone category theory, to readers who already know Kan extensions and are looking for familiar examples—or unusual ones.

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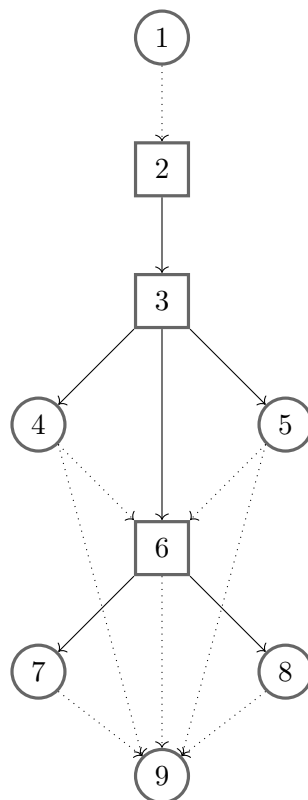
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How to read this, and why

Here are some suggested routes through the text, depending on your background.

1. **If you have not learned about sets and maps**, you can still get a feel for the subject by browsing Sections 1 and 4, which mainly concern functions from the real numbers to the real numbers. Much of the rest will be out of reach, though. I recommend starting with a suitable introductory mathematics text.
2. **If you know about sets and maps but little other modern mathematics**, you can read the whole text apart from a few examples. Section 2 gives a crash course in the order theory we need, but you may also want to consult a textbook for a more leisurely account. Feel free to skip Sections 5 and 8 if they are too difficult, and Section 7 if you do not yet see the point of it.
3. **If you are studying mathematics, whether or not you have studied category theory**, you can read everything. You can skip the crash course in Section 2, checking only the notation and terminology.
4. **If you already know Kan extensions**, this text is for you too. Of course, you can read everything, and some of the examples were chosen especially with you in mind.

The dependencies between sections are shown below. Circles mark discussion and examples; squares mark theory. Solid arrows indicate logical dependencies, and dotted arrows indicate dependencies in motivation.



Here is a guide to the sections. Sections 1 and 9 are discussion, Sections 2, 3, and 6 are theory, and Sections 4, 5, 7, and 8 are examples. The examples are the heart of this text, but they are also the parts I expect you to skip selectively, depending on your background and interests.

Section 1 is for everyone. Using the extension of the exponential function, it explains why posets have anything to do with extension problems, and gives us a reason to care about what follows. Section 2 is a crash course in the order theory we need, for readers who have not met posets before. Section 3 is for everyone: it defines our main topic, Kan extensions between posets. Section 4 is also for everyone. Its examples are mostly at high school level, such as functions from the reals to the reals, and introduce basic properties of Kan extensions. Section 5 is for readers studying mathematics. It presents examples involving topological spaces, measure spaces, and other subjects from an undergraduate mathematics course. By drawing on several fields, I hope to give you a feel for just how many places Kan extensions turn up. Section 6 introduces a formula for computing Kan extensions, for everyone. It gives a single way to compute every Kan extension we have met up to that point. Section 7 offers exercises on the theoretical properties of Kan extensions for particularly keen readers. These are poset versions of general theorems about Kan extensions in category theory, with proofs that work just as in the general case. I hope they provide a useful refresher if you already know Kan extensions, and a useful preview if you plan to study them in category theory later. Section 8 is for everyone who knows the formula from Section 6. It presents some slightly unusual Kan extensions. Section 9 briefly introduces Kan extensions in category theory and says something about the role they play.

1 Introduction: What is a natural extension?

We often want to extend a function that is only partially defined. For example, in high school mathematics we extended the exponential function $2^q : \mathbb{Q} \rightarrow \mathbb{R}$, defined on the rational numbers, to a function $2^r : \mathbb{R} \rightarrow \mathbb{R}$ on the real numbers. In a diagram, we are looking for the dotted arrow

below. Here ι is simply the inclusion map.

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{2^q} & \mathbb{R} \\ & \searrow \iota & \nearrow \text{dotted} \\ & \mathbb{R} & \end{array}$$

For the usual extension, we take a sequence of rational numbers $q_1, q_2, \dots \rightarrow r$ converging to a real number r , and define $2^r := \lim_{n \rightarrow \infty} 2^{q_n}$. Notice that this is not the only possible extension. Indeed, if a function f makes the diagram

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{2^q} & \mathbb{R} \\ & \searrow \iota & \nearrow f \\ & \mathbb{R} & \end{array}$$

commute—that is, if it extends 2^q —we can change its values at irrational numbers however we like and it will still satisfy the condition. There are therefore countless such functions f . The usual extension 2^r is just one of them. So, unfortunately, **there is no natural extension of a map**.¹

To say that the usual extension 2^r is natural, we need more than just the underlying sets $\mathbb{Q}$ and $\mathbb{R}$. One solution is to take their order into account as well. If we allow only order-preserving (that is, nondecreasing) functions f , then the usual extension 2^r is the unique extension.²

Thus, if we want natural extensions, sets and maps alone are not enough. Taking the relations between elements into account may give us a natural extension.

The theory of posets and order-preserving maps looks like a promising setting for our search. **Perhaps natural extensions do exist in order theory!**

Remark 1.1 (Category theory). To work with relationships between elements richer than an order, we can consider categories, which generalize posets. Kan extensions are usually discussed in category theory (at least as far as I know in November 2021). We will say a little more about this in Section 9.

2 A crash course in order theory

Here are the basic definitions from order theory.

Definition 2.1 (Poset). A *poset*, or partially ordered set, is a pair $(X, \leq)$ consisting of a set X and a binary relation $\leq$ satisfying the following three conditions:

- $x \leq x$ for every $x \in X$.
- If $x \leq y$ and $y \leq z$, then $x \leq z$, for all $x, y, z \in X$.
- If $x \leq y$ and $y \leq x$, then $x = y$, for all $x, y \in X$.

These three conditions are called reflexivity, transitivity, and antisymmetry, respectively.

Example 2.2 (Integers, rationals, and reals). The set $\mathbb{Z}$ of integers, the set $\mathbb{Q}$ of rational numbers, and the set $\mathbb{R}$ of real numbers are posets under the usual order $\leq$.

Example 2.3 (Power sets). For any set A , its power set $\mathcal{P}(A)$ (the set of all its subsets) is a poset under inclusion $\subset$.

¹We will make the mathematical statement precise in Example 4.4, after defining Kan extensions.

²The definition of 2^r in terms of limits suggests another solution: require f to be continuous. Order has the advantage of appearing just about everywhere, so it gives us a very general setting. In fact, even the idea of preserving limits will underlie the formula for computing Kan extensions that we will see later!

Example 2.4 (The trivial order). Any set X becomes a poset if we put $x \leq y \iff x = y$ for all $x, y \in X$. This is called the trivial order.

Exercise 2.5 (Divisibility). Check that the nonnegative integers $\mathbb{N}$ form a poset under divisibility $|$.

Exercise 2.6 (Human relations!?). Here is a less-than-rigorous exercise. Let P be the set of people. Define a binary relation $\leq$ on P by giving $p \leq q$ each of the following meanings. Which ones are orders? For those that are not, say which of reflexivity, transitivity, and antisymmetry fail.

1. p is a descendant of q .
2. p is a descendant of q , or $p = q$.
3. p and q know each other.
4. p likes q .
5. Starting from p , we can reach q by following a finite chain of acquaintances.
6. p and q are the same age.

From now on, we will write a poset $(P, \leq)$ simply as P , leaving the relation understood. This is one of those convenient abuses of notation that mathematics is full of.

Definition 2.7 (Order-preserving map). A map $f : P \rightarrow Q$ between posets is *order-preserving* if, for all $x, y \in P$,

$$x \leq y \implies f(x) \leq f(y)$$

holds.

Example 2.8 (Over the reals). A map $f : \mathbb{R} \rightarrow \mathbb{R}$ is order-preserving exactly when it is nondecreasing. The functions $f(x) = x$, $f(x) = 0$, and $f(x) = 2^x$ are order-preserving, but $f(x) = -x$, $f(x) = 2^{-x}$, and $f(x) = \sin(x)$ are not.

Definition 2.9 (Comparing order-preserving maps). For order-preserving maps $f, g : P \rightarrow Q$ between posets, we define $f \leq g$ to mean

$$f(p) \leq g(p) \quad \text{for every } p \in P.$$

Example 2.10 (Over the reals). If $f(x) = x + 2$ and $g(x) = x + 3$, then $f \leq g$. But if $f(x) = 2^x$ and $g(x) = 3^x$, neither $f \leq g$ nor $g \leq f$ holds.

Exercise 2.11 (The poset of order-preserving maps). For posets P, Q , check that the set Q^P of all order-preserving maps from P to Q is itself a poset under the order $\leq$ just defined.

Definition 2.12 (Least and greatest elements). An element p of a poset P is its *least element* if

$$p \leq q \quad \text{for every } q \in P,$$

and its *greatest element* if

$$q \leq p \quad \text{for every } q \in P.$$

Example 2.13 (Over the reals). The poset $\mathbb{R}$ has neither a greatest nor a least element.

Example 2.14 (Nonnegative integers). The least element of $\mathbb{N}$ is 0. There is no greatest element.

Least and greatest elements are opposite notions. We call such opposite notions *dual*. Let us be a little more precise about what duality means here. If $(P, \leq)$ is a poset, then reversing its order gives another poset $(P, \geq)$. A notion C in this reversed poset is called the dual of the notion C in P .

What we have covered so far in Section 2 is enough to read through Section 5, apart from a few examples, and also Section 9. For Sections 6, 7, and 8, we will need the following ideas.

Definition 2.15 (Upper and lower bounds). Let $S \subset P$ be a subset of a poset. An element $p \in P$ is an *upper bound* of S if

$$s \leq p \quad \text{for every } s \in S.$$

It is a *lower bound* of S if

$$p \leq s \quad \text{for every } s \in S.$$

Upper and lower bounds are dual notions too. Notice that every element is both an upper bound and a lower bound of the empty set.

Definition 2.16 (sup, inf). For a subset S of a poset P , the *supremum* $\sup S$ is the least upper bound of S . The *infimum* $\inf S$ is its greatest lower bound.

Suprema and infima are dual too. The idea of taking the least upper bound or greatest lower bound actually goes back to something we learned in elementary school.

Example 2.17 (Least common multiples and greatest common divisors). In the poset $(\mathbb{N}, |)$ of nonnegative integers ordered by divisibility, the supremum of $\{12, 18, 30\}$ is 180 and its infimum is 6. In general, the supremum of $S \subset \mathbb{N}$ is the least common multiple of its elements, and its infimum is their greatest common divisor.

Example 2.18 (Unions and intersections). For a set A , consider its power set $\mathcal{P}(A)$ (the set of all its subsets), ordered by inclusion $\subset$. For a subset $S \subset \mathcal{P}(A)$ —a set of subsets of A !—the supremum is the union of the members of S , and the infimum is their intersection.

Example 2.19 (Over the reals). The subset $S = \{x \in \mathbb{Q}(\subset \mathbb{R}) \mid x^2 \leq 2\}$ of $\mathbb{R}$ has supremum $\sqrt{2}$ and infimum $-\sqrt{2}$. On the other hand, regarded as a subset of $\mathbb{Q}$, the set $S = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$ has neither a supremum nor an infimum.

Remark 2.20 (The category-theoretic counterparts). For readers who know category theory, here is the correspondence between the notions from Section 2 and their category-theoretic counterparts.

$$\begin{aligned} \text{posets} &\leftrightarrow \text{categories} \\ \text{order-preserving maps} &\leftrightarrow \text{functors} \\ \text{inequalities between order-preserving maps} &\leftrightarrow \text{natural transformations} \\ \text{least and greatest elements} &\leftrightarrow \text{initial and terminal objects} \\ \text{suprema and infima} &\leftrightarrow \text{colimits and limits} \end{aligned}$$

When we restrict attention to categories that are posets, the category-theoretic notions on the right are exactly the order-theoretic notions on the left.

3 Defining Kan extensions

We begin this section with the definition of a Kan extension. In the definition, hg denotes the composite of the maps g and h .

Definition 3.1 (Kan extensions). Let P, Q, R be posets, and let $f : P \rightarrow R$ and $g : P \rightarrow Q$ be order-preserving maps. The *left Kan extension of f along g* is the least order-preserving map $h : Q \rightarrow R$ satisfying $f \leq hg$:

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g \quad \wedge | \quad \nearrow h & \\ & Q & \end{array}$$

We write this map as $\text{Lan}_g f : Q \rightarrow R$. The *right Kan extension of f along g* is the greatest order-preserving map $h : Q \rightarrow R$ satisfying $f \geq hg$:

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g \quad \vee | \quad \nearrow h & \\ & Q & \end{array}$$

We write this map as $\text{Ran}_g f : Q \rightarrow R$.

Less formally, a left Kan extension h must be large enough for hg to bound f from above, so that $f \leq hg$. Among all the maps that meet this requirement, we choose the least one. Dually, a right Kan extension h must be small enough for hg to bound f from below, so that $f \geq hg$. Among all the maps that meet this requirement, we choose the greatest one. Of course, we need examples to unpack this definition. We will get to those in the next section; first, a few remarks.

Equivalently, the left Kan extension of f along g is an order-preserving map $\text{Lan}_g f : Q \rightarrow R$ satisfying

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g \quad \wedge | \quad \nearrow \text{Lan}_g f & \\ & Q & \end{array}$$

and, for every order-preserving map $h : Q \rightarrow R$,

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g \quad \wedge | \quad \nearrow h & \\ & Q & \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g \quad \wedge | \quad \nearrow \text{Lan}_g f & \\ & Q & \end{array} \quad \begin{array}{c} \nearrow h \\ \nwarrow \end{array}$$

as shown above.

Since Kan extensions are defined as greatest or least elements, they are unique whenever they exist.³ So a definition of the form “define this to be the left Kan extension of that along such-and-such” is unambiguous as soon as we have proved existence.

Kan extensions need not exist. We will see examples later. To put a positive spin on this, perhaps we can say that **when we do not want them to exist, they have the good sense not to**. I hope the examples in the next section will persuade you to share this optimistic view.

The definition does not require a Kan extension to be an actual extension: we do not require $f = (\text{Lan}_g f) \circ g$. As we will see, a Kan extension generally need not be an actual extension.⁴ In this sense, the inequality $f \leq (\text{Lan}_g f) \circ g$ can be viewed as an **unavoidable discrepancy**. On the bright side, **Kan extensions find the best approximation to an extension**, even when an actual extension does not exist.

Notice also that, as so often in order theory and category theory, there are two dual extensions: the left and the right Kan extension. Let us turn to examples.

Remark 3.2 (Relation to Kan extensions in category theory). Kan extensions between posets are not merely analogues of Kan extensions in category theory: they are special cases.

³In category theory, as with other universal properties, they are unique up to canonical isomorphism whenever they exist.

⁴There is a useful sufficient condition for it to be one.

4 Easier toys

In this section, we unpack the definition through simple, intuitive examples, mostly involving real and rational numbers, to get comfortable with the idea and properties of Kan extensions. Do try to check that these really are Kan extensions, but if that feels difficult, there is no need to worry too much just yet. Every Kan extension outside Section 8 can be checked easily using the formula in Section 6.

Back to our motivating example

Example 4.1 (Extending the exponential function). Let us check that the exponential function from Section 1 really is a Kan extension. In Definition 3.1, take P to be the poset $\mathbb{Q}$ of rational numbers, and take both Q and R to be $\mathbb{R}$. We extend $2^q : \mathbb{Q} \rightarrow \mathbb{R}$ along the inclusion $\iota : \mathbb{Q} \rightarrow \mathbb{R}$.

The left Kan extension is the least order-preserving map $h : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

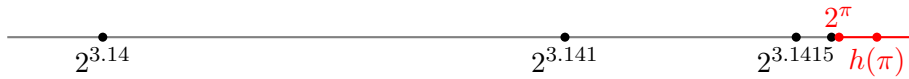
$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{2^q} & \mathbb{R} \\ & \searrow \iota \quad \wedge \mid \quad \nearrow h & \\ & \mathbb{R} & \end{array}$$

or, in other words, $2^q \leq h(q)$ for every $q \in \mathbb{Q}$. Let us see why this is the usual function $2^r : \mathbb{R} \rightarrow \mathbb{R}$. Any order-preserving map $h : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $2^q \leq h(q)$ for every rational q must lie above $2^r : \mathbb{R} \rightarrow \mathbb{R}$.

Rather than write out a detailed proof, let us convince ourselves that $2^\pi \leq h(\pi)$. Take the sequence of rational numbers $3, 3.1, 3.14, 3.141, 3.1415, \dots$ converging to π from below. Then

$$\begin{aligned} 2^3 &\leq h(3) \leq h(\pi) \\ 2^{3.1} &\leq h(3.1) \leq h(\pi) \\ 2^{3.14} &\leq h(3.14) \leq h(\pi) \\ 2^{3.141} &\leq h(3.141) \leq h(\pi) \\ 2^{3.1415} &\leq h(3.1415) \leq h(\pi) \\ &\vdots \end{aligned}$$

which shows that $2^\pi \leq h(\pi)$.



A careful version of this argument confirms that the usual function $2^r : \mathbb{R} \rightarrow \mathbb{R}$ is the left Kan extension:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{2^q} & \mathbb{R} \\ & \searrow \iota \quad \wedge \mid \quad \nearrow 2^r = \text{Lan}_\iota 2^q & \\ & \mathbb{R} & \end{array}$$

Similarly, the right Kan extension is also 2^r . In this case, the two dual extensions—the left and right Kan extensions—coincide. In Section 1, we saw that trying to extend a function merely as a map of sets did not give us a natural extension. Things worked out once we discussed extensions in the language of posets and order-preserving maps, with inequalities $\leq$ (or, more generally, arrows in a category), rather than in the language of sets and maps, with equalities $=$!

When left and right Kan extensions differ

Many readers already know an example where the two dual extensions do not coincide: floor and ceiling functions.

Example 4.2 (Floor and ceiling: approximating reals by integers). This time, let us extend $\text{id}_{\mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{Z}$ along the inclusion $\iota : \mathbb{Z} \rightarrow \mathbb{R}$. In other words, what is a natural way to turn a real number into an integer? Or: what is a natural integer approximation to a real number?

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\text{id}_{\mathbb{Z}}} & \mathbb{Z} \\ & \searrow \iota & \nearrow h \\ & \mathbb{R} & \end{array}$$

The least map h satisfying this diagram—the left Kan extension of $\text{id}_{\mathbb{Z}}$ along ι —is the floor function $\lfloor \cdot \rfloor$. Dually, the right Kan extension is the ceiling function $\lceil \cdot \rceil$.



Thus, **left and right Kan extensions need not coincide**.

Exercise 4.3 (Maximal extensions of real-valued functions). For $r \in \mathbb{R}$, define an order-preserving map $s_r : \mathbb{Q} \rightarrow \mathbb{R}$ by

$$s_r(q) := \begin{cases} 1 & (q \geq r) \\ 0 & (q < r) \end{cases}$$

and consider its left and right Kan extensions $\text{Lan}_{\iota} s_r, \text{Ran}_{\iota} s_r$ along the inclusion $\iota : \mathbb{Q} \rightarrow \mathbb{R}$.

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{s_r} & \mathbb{R} \\ & \searrow \iota & \nearrow \text{Lan}_{\iota} s_r \\ & \mathbb{R} & \end{array} \quad \begin{array}{ccc} \mathbb{Q} & \xrightarrow{s_r} & \mathbb{R} \\ & \searrow \iota & \nearrow \text{Ran}_{\iota} s_r \\ & \mathbb{R} & \end{array}$$

1. Show that both Kan extensions exist.
2. Find a necessary and sufficient condition on $r \in \mathbb{R}$ for them to coincide: $\text{Lan}_{\iota} s_r = \text{Ran}_{\iota} s_r$.

When Kan extensions do not exist

A Kan extension need not exist in the first place. Here are a few examples of nonexistence.

Example 4.4 (There is no natural extension of a map). In Section 1, we claimed that “there is no natural extension of a map.” Here is a precise formulation of that intuition. Take a set X and a subset S . We want to extend a map $f : S \rightarrow Y$ to all of X . Regard the sets as posets with the trivial order. The (left) Kan extension of f along the inclusion $\iota : S \rightarrow X$,

$$\begin{array}{ccc} S & \xrightarrow{f} & Y \\ & \searrow \iota & \nearrow ? \\ & X & \end{array}$$

does not exist, except in the trivial cases where $S = X$ or Y has one element.

Exercise 4.5 (There is no natural rational approximation to a real number). Following Example 4.2, consider the Kan extension of $\text{id}_{\mathbb{Q}} : \mathbb{Q} \rightarrow \mathbb{Q}$ along the inclusion $\iota : \mathbb{Q} \rightarrow \mathbb{R}$. We are asking for a natural rational approximation to a real number.

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\text{id}_{\mathbb{Q}}} & \mathbb{Q} \\ & \searrow \iota \quad \wedge \downarrow \quad \nearrow ? & \\ & \mathbb{R} & \end{array}$$

1. For a positive integer $n \geq 1$, define $h_n : \mathbb{R} \rightarrow \mathbb{Q}$ by $h_n(x) := \lceil nx \rceil / n$. Show that h_n is order-preserving and satisfies

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\text{id}_{\mathbb{Q}}} & \mathbb{Q} \\ & \searrow \iota \quad \wedge \downarrow \quad \nearrow h_n & \\ & \mathbb{R} & \end{array}$$

2. Show that there is no natural rational approximation to a real number: the left Kan extension $\text{Lan}_{\iota} \text{id}_{\mathbb{Q}}$ of $\text{id}_{\mathbb{Q}}$ along ι does not exist.

Dependence on the order

Naturally, the order is essential to the definition of a Kan extension. Here is a simple example in which changing the order changes the Kan extension.

Example 4.6 (Dirichlet's function). Define an order-preserving map $c_1 : \mathbb{Q} \rightarrow [0, 1] := \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ by $c_1(q) = 1$: the constant function with value 1. Its left Kan extension along the inclusion $\iota : \mathbb{Q} \rightarrow \mathbb{R}$,

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{c_1} & [0, 1] \\ & \searrow \iota \quad \wedge \downarrow \quad \nearrow \text{Lan}_{\iota} c_1 & \\ & \mathbb{R} & \end{array}$$

is the constant function $(\text{Lan}_{\iota} c_1)(x) = 1$ from $\mathbb{R}$ to $[0, 1]$. But if we leave the usual order on $[0, 1]$ and give $\mathbb{Q}$ and $\mathbb{R}$ the trivial order, then $\text{Lan}_{\iota} c_1$ becomes Dirichlet's function:

$$(\text{Lan}_{\iota} c_1)(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \notin \mathbb{Q}) \end{cases}$$

Having just said that the order matters, I may be about to confuse you: for Kan extensions between posets, the order on P (the poset at the very start) is not actually essential.⁵ Here is an exercise to show this.

Exercise 4.7 (Independence of the order on P). Given a Kan extension between posets,

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g \quad \wedge \downarrow \quad \nearrow \text{Lan}_g f & \\ & Q & \end{array}$$

show that $\text{Lan}_g f$ remains the left Kan extension if we change the order on P in any way that keeps f and g order-preserving. In particular, we can give P the trivial order.

⁵It does matter for Kan extensions between general categories. Here we are working with orders, so every diagram commutes when viewed categorically, and the naturality condition for a natural transformation is automatic.

Extensions that are not really extensions

So far, every Kan extension we have met has been an actual extension. In other words, the inequality $f \leq (\text{Lan}_g f) \circ g$ representing the unavoidable discrepancy has been an equality. Now let us look at Kan extensions that are not actual extensions. We will also see that the unavoidable discrepancy $f \leq (\text{Lan}_g f) \circ g$ takes the form of familiar inequalities.

Example 4.8 (The quantifiers $\exists, \forall$). Define a poset Ω on the set $\Omega = \{\text{true}, \text{false}\}$ by $\text{false} \leq \text{true}$. Equivalently, these are the truth values ordered by logical implication $\Rightarrow$ (“implies”). Give a set X the trivial order. Every map $X \rightarrow \Omega$ is then order-preserving. Thus order-preserving maps $X \rightarrow \Omega$ correspond bijectively to assignments of either true or false to each element. With this in mind, we call an order-preserving map $X \rightarrow \Omega$ a predicate on X . For sets X, Y , a predicate on $X \times Y$ can be viewed as a predicate in two variables.

Is there a natural way to reduce the number of variables of a predicate? That is, given a predicate $\phi : X \times Y \rightarrow \Omega$, does it have a Kan extension along the first projection $\pi_1 : X \times Y \rightarrow X$?

$$\begin{array}{ccc} X \times Y & \xrightarrow{\phi} & \Omega \\ & \searrow \pi_1 \quad \swarrow ? & \\ & X & \end{array} \quad \wedge \mid$$

In fact,

$$x \mapsto \exists y \in Y, \phi(x, y)$$

is the left Kan extension. We denote it by $\exists y \in Y, \phi(-, y)$. The statement $\exists y \in Y, \phi(x, y)$ means that there is some $y \in Y$ for which $\phi(x, y)$ is true. The unavoidable discrepancy $\phi \leq (\exists y \in Y, \phi(-, y)) \circ \pi_1$ is, at each $(x_0, y_0) \in X \times Y$,

$$\phi(x_0, y_0) \implies \exists y \in Y, \phi(x_0, y)$$

where $\implies$ means logical implication. In words, the discrepancy says: if the pair (x_0, y_0) satisfies ϕ , then there is a y for which (x_0, y) satisfies ϕ . Just take $y = y_0$!

This left Kan extension is an actual extension precisely when the truth of the two-variable predicate ϕ depends only on the first variable, not the second. Dually, $x \mapsto \forall y \in Y, \phi(x, y)$ is the right Kan extension, and its unavoidable discrepancy is, at each $(x_0, y_0) \in X \times Y$,

$$\forall y \in Y, \phi(x_0, y) \implies \phi(x_0, y_0)$$

The two natural ways to remove a variable from a predicate are $\exists$ and $\forall$. For more on this example, see Section 9.5 of [4] or Section I.9 of [3].

Suprema and infima, which will play an essential role from Section 6 onward, are also important examples of Kan extensions that are generally not actual extensions.

Example 4.9 (Suprema and infima). Write $\mathbf{1}$ for the poset with exactly one element. By looking at the image, (order-preserving) maps from $\mathbf{1}$ to a (partially ordered) set R correspond bijectively to elements of R . For a poset R , a subset $S \subset R$, and an element $r \in R$, saying that r is the supremum of S is equivalent to saying that its corresponding (order-preserving) map $[r] : \mathbf{1} \rightarrow R$ is the left Kan extension of the inclusion $\iota : S \rightarrow R$ along the unique (order-preserving) map $! : S \rightarrow \mathbf{1}$:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & R \\ & \searrow ! \quad \swarrow [r] & \\ & \mathbf{1} & \end{array} \quad \wedge \mid$$

Dually, the infimum can be defined as a right Kan extension. Thus **the two natural ways to obtain an element of a poset from a subset are sup and inf**. The unavoidable discrepancies are, for each $s \in S$, $s \leq \sup S$ and $\inf S \leq s$. This Kan extension is an actual extension precisely when S has at most one element.

Example 4.10 (Inverse images). For a map $f : X \rightarrow Y$, consider the order-preserving direct-image map $f(-) : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$. Here the order is inclusion.

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \\ & \searrow f(-) \quad \wedge | \quad \nearrow f^{-1} & \\ & \mathcal{P}(Y) & \end{array}$$

The inverse-image map f^{-1} is the left Kan extension of $\text{id}_{\mathcal{P}(X)}$ along $f(-)$. The unavoidable discrepancy is the inclusion

$$S \subset f^{-1}(f(S))$$

for each $S \subset X$. In other words, elements of X that have the same image under f become indistinguishable. This left Kan extension is an actual extension precisely when f is injective.

5 Harder toys

For some readers, here are examples that need a little more background. In these subjects, Kan extensions really come into their own when we work with general categories, but we can already find examples among posets.

Topological spaces

Example 5.1 (Interiors). For a topological space X , consider its power set $\mathcal{P}(X)$ and its set of open subsets $\mathcal{O}(X)$, both ordered by inclusion. What is the left Kan extension of $\text{id}_{\mathcal{O}(X)}$ along the inclusion ι ? In other words, what is the best open-set approximation to a subset?

$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{\text{id}_{\mathcal{O}(X)}} & \mathcal{O}(X) \\ & \searrow \iota \quad \wedge | \quad \nearrow \text{Int} & \\ & \mathcal{P}(X) & \end{array}$$

As you would expect, the interior operator $\text{Int} : \mathcal{P}(X) \rightarrow \mathcal{O}(X)$ is the Kan extension we want.

Let us leave the right Kan extension as an exercise.

Exercise 5.2 (Right Kan extension to the family of open sets). In the setting of Example 5.1, prove the following.

1. The right Kan extension exists too.
2. The left and right Kan extensions coincide if and only if X is a T_1 space.

Set operations

For the general theory and category-theoretic background to these examples, see [6].

Exercise 5.3 (Various closure operators). Take a set X , its power set $\mathcal{P}(X)$, and a subposet $\mathcal{B}$ of $\mathcal{P}(X)$. Consider the right Kan extension of $\text{id}_{\mathcal{B}}$ along the inclusion $\iota : \mathcal{B} \rightarrow \mathcal{P}(X)$.⁶

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\text{id}_{\mathcal{B}}} & \mathcal{B} \\ & \searrow \iota \quad \vee | \quad \nearrow f & \\ & \mathcal{P}(X) & \end{array}$$

⁶It need not exist in general.

1. If X is a vector space (or a group, monoid, lattice, ring, or another algebraic structure) and $\mathcal{B}$ is the set of all linear subspaces (or the corresponding subalgebras), show that taking the generated subspace (or subalgebra) defines the right Kan extension.
2. If $X = \mathbb{R}^2$ and $\mathcal{B}$ is the set of convex subsets, show that taking the convex hull defines the right Kan extension.
3. If X is a topological space and $\mathcal{B}$ is its set of closed subsets, show that the closure operator defines the right Kan extension.

Exercise 5.4 (Existential quantification and direct images). A subset $S \subset X$ corresponds to a predicate $\chi_S : X \rightarrow \Omega$ by

$$\chi_S(x) = \text{true} \iff x \in S.$$

In Example 4.8, replace the projection $\pi_1 : X \times Y \rightarrow X$ by a general map $g : X \rightarrow Y$.

1. Check that, in terms of the corresponding subsets, the left Kan extension is the direct image.

$$\begin{array}{ccc} X & \xrightarrow{\chi_S} & \Omega \\ & \searrow g \quad \wedge | \quad \nearrow \chi_{g(S)} & \\ & Y & \end{array}$$

2. Write down the subset corresponding to the right Kan extension.⁷

$$\begin{array}{ccc} X & \xrightarrow{\chi_S} & \Omega \\ & \searrow g \quad \vee | \quad \nearrow \chi_{g_1(S)} & \\ & Y & \end{array}$$

Measure theory

Several important definitions in measure theory can be expressed in terms of Kan extensions.

Example 5.5 (Integration on a measure space). One essential step in defining integration on a measure space was to extend the integral from nonnegative simple functions to nonnegative measurable functions. This, too, is a Kan extension problem. For a measure space $(X, \mathcal{B}, \mu)$, let Mea_X be the poset of nonnegative measurable functions, ordered pointwise. The integral $\int d\mu : \text{Mea}_X \rightarrow [0, \infty]$ is the left Kan extension of the integral $\int d\mu : \text{Sim}_X \rightarrow [0, \infty]$ on nonnegative simple functions, along the inclusion $\iota : \text{Sim}_X \rightarrow \text{Mea}_X$ of their pointwise-ordered poset.

$$\begin{array}{ccc} \text{Sim}_X & \xrightarrow{\int d\mu} & [0, \infty] \\ & \searrow \iota \quad \wedge | \quad \nearrow \int d\mu & \\ & \text{Mea}_X & \end{array}$$

When asking for a Kan extension of $f : P \rightarrow R$, we choose the map $g : P \rightarrow Q$ along which to extend. In particular, we choose the domain Q of $\text{Lan}_g f : Q \rightarrow R$. This seems rather ill-suited to extension problems where we want “the largest possible domain.” But we can also try to **use both left and right Kan extensions to obtain the largest possible domain**. More concretely, we restrict to the part where the two Kan extensions agree, and regard this as the maximal extension.⁸ The natural notions of extension come as a dual pair; if we insist on having just one, we can take their common part.

⁷This is called the dual image on the nLab: <https://ncatlab.org/nlab/show/internal+logic>.

⁸For readers who know category theory: the completion of a measure μ is the equalizer of $\text{Lan}_\iota \mu$ and $\text{Ran}_\iota \mu$ in Cat . We can also use Poset or Set , though this gives a less general perspective.

Example 5.6 (Completion of a measure space). For a measure space $(X, \mathcal{B}, \mu)$, the map $\mu : \mathcal{B} \rightarrow [0, \infty]$ is order-preserving. We want to extend it to as much of the power set $\mathcal{P}(X)$ as possible. First, take its left and right Kan extensions along the inclusion $\iota : \mathcal{B} \rightarrow \mathcal{P}(X)$.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\mu} & [0, \infty] \\ & \searrow \iota & \nearrow \text{Lan}_\iota \mu \\ & \mathcal{P}(X) & \nearrow \text{Ran}_\iota \mu \end{array}$$

Let $\bar{\mathcal{B}} \subset \mathcal{P}(X)$ be the part where $\text{Lan}_\iota \mu$ and $\text{Ran}_\iota \mu$ agree, and denote their common restriction by $\bar{\mu} : \bar{\mathcal{B}} \rightarrow [0, \infty]$.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\mu} & [0, \infty] \\ & \searrow \iota & \nearrow \text{Lan}_\iota \mu \\ & \mathcal{P}(X) & \nearrow \text{Ran}_\iota \mu \\ & \nwarrow \bar{\mu} & \nearrow \bar{\mu} \\ \bar{\mathcal{B}} & & \end{array}$$

If $(X, \mathcal{B}, \mu)$ is finite, meaning $\mu(X) < \infty$,⁹ then $(X, \bar{\mathcal{B}}, \bar{\mu})$ is its completion.

We will postpone the exercise of checking this measure-space example until Section 6, when we have the formula. For now, here is something we can check directly from the definitions.

Exercise 5.7 (The maximal extension of a measure is complete). Consider $(X, \bar{\mathcal{B}}, \bar{\mu})$ as defined in Example 5.6. If $N \in \bar{\mathcal{B}}$ and $\bar{\mu}(N) = 0$, show that every subset $S \subset N$ belongs to $\bar{\mathcal{B}}$ and satisfies $\bar{\mu}(S) = 0$.

6 A formula for computing Kan extensions

Almost every Kan extension in this text, including all the examples so far, can be constructed in essentially the same way, using Theorem 6.1 below. To see the idea, let us return to the exponential function:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{2^q} & \mathbb{R} \\ & \searrow \iota & \nearrow 2^r = \text{Lan}_\iota 2^q \\ & \mathbb{R} & \end{array}$$

As we recalled in Section 1, its natural extension 2^r (both the left and the right Kan extension) is defined by choosing a rational sequence $q_1, q_2, \dots \rightarrow r$ and taking the limit of $2^{q_1}, 2^{q_2}, \dots$. For $r = \pi$, for instance,

$$\begin{array}{ccc} 3, 3.1, 3.14, 3.141, \dots & \rightarrow & \pi \\ & \Downarrow & \\ 2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, \dots & \rightarrow & 2^\pi \end{array}$$

is one way to do it. This definition uses limits of sequences, but let us rewrite it in the language of posets:

$$\begin{aligned} 2^\pi &:= \sup_{q \leq \pi} 2^q \\ & (= \sup \{ 2^q \in \mathbb{R} \mid q \in \mathbb{Q} \wedge q \leq \pi \}) \end{aligned}$$

⁹For a nonfinite measure space, this is still almost the completion, but things go wrong at ∞ : too many sets become measurable. If $\bar{\mu}(S) = \infty$, then $\bar{\mu}(S') = \infty$ for every larger set $S \subset S'$. This looks reasonable at first, but $\bar{\mathcal{B}}$ may lose its σ -additivity.

Here q ranges over all rational numbers at most π . Instead of choosing one rational sequence converging to π , we take all rationals at most π .

This construction generalizes to posets.

Theorem 6.1 (A formula for Kan extensions between posets). Let P, Q, R be posets, and let $f : P \rightarrow R$ and $g : P \rightarrow Q$ be order-preserving maps.

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g & \\ & & Q \end{array}$$

If $\sup_{g(p) \leq q} f(p)$ exists in R for every $q \in Q$, then

$$q \mapsto \sup_{g(p) \leq q} f(p)$$

gives the left Kan extension of f along g .

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g & \nearrow \sup_{g(p) \leq -} f(p) \\ & & Q \end{array} \quad \wedge \mid$$

Proof.

Write $l : Q \rightarrow R$ for the map $q \mapsto \sup_{g(p) \leq q} f(p)$ in the statement.

First, l is order-preserving. If $q \leq q'$, then $\{f(p) \mid p \in P \wedge g(p) \leq q\} \subset \{f(p) \mid p \in P \wedge g(p) \leq q'\}$. Taking suprema gives $l(q) \leq l(q')$.

Next, we show $f \leq lg$. Take any $p \in P$. The trivial (in)equality $g(p) \leq g(p)$ gives

$$f(p) \leq \sup_{g(p') \leq g(p)} f(p') = l(g(p)).$$

Finally, suppose $h : Q \rightarrow R$ is order-preserving and $f \leq hg$. We must show $l \leq h$. Take any $q \in Q$. By the definition of supremum, it is enough to show that $h(q)$ is an upper bound of $\{f(p) \mid p \in P \wedge g(p) \leq q\}$. For any $p \in P$ with $g(p) \leq q$, order preservation gives $hg(p) \leq h(q)$, while our assumption gives $f(p) \leq hg(p)$. Together these give $f(p) \leq h(q)$. $\square$

Exercise 6.2 (A formula for right Kan extensions). Show, dually, that right Kan extensions can be computed using infima whenever the required infima exist.

Let me try to explain this formula a little less formally. When we ask for a (left) Kan extension, we start with

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g & \\ & & Q \end{array}$$

and want to extend to a map $Q \rightarrow R$: we want to take an element q of Q and return an element of R . Somehow we want to go through P , of course, but g points the wrong way for us to do that directly.

So here is what the formula does: take an element of Q , **make a mould** of it out of elements of P , and then send that mould into R ! More concretely, given $q \in Q$, form the subset

$$\{p \in P \mid g(p) \leq q\} \subset P$$

of P . This collects the elements of P that lie below q , via g . That is our “mould.” We then use it to produce an element of R : send the mould through f to get

$$\{f(p) \mid p \in P, g(p) \leq q\} \subset R$$

and take its supremum. This process of making a mould and reconstructing from it, as we move through Q, P, R , is one way to understand what the formula

$$q \mapsto \sup_{g(p) \leq q} f(p)$$

is really doing!

To labour the point a little: taking the supremum of all elements below an element would ordinarily just reconstruct that element. The formula performs this reconstruction while moving between posets, and thereby gives a natural extension. Try this out on the following trivial extension problem.

Exercise 6.3 (Educational overkill). For a poset P , use Theorem 6.1 to check again that the left Kan extension of id_P along id_P is id_P .

$$\begin{array}{ccc} P & \xrightarrow{\text{id}_P} & P \\ & \searrow \text{id}_P \quad \wedge \mid \quad \nearrow \text{id}_P & \\ & P & \end{array}$$

The next exercise, together with Exercise 8.5, is one of the most important in this text. Please do give it a go.

Exercise 6.4 (All the examples so far). Every Kan extension we have met so far can be computed by this formula. Check this for as many examples as you can.

I hope this exercise has given you some sense of the formula’s power. As we will mention in Section 9, its scope becomes vastly greater—incomparably so!—when we move to Kan extensions in category theory.

Definition 6.5 (Pointwise Kan extensions). A Kan extension computed by Theorem 6.1 is called a *pointwise Kan extension*.

Remark 6.6 (Kan extensions beyond the formula). As Example 8.1 will show, some Kan extensions are not pointwise.

But there is no need to be too pessimistic. As the fact that all our examples so far were pointwise suggests, there is a rule of thumb (not a mathematical claim!) that **the important Kan extensions are pointwise**. Section 6.3 of [1] says:

By the categorical community’s consensus, the important Kan extensions are pointwise Kan extensions.

as quoted above. Indeed, [5] calls what we call Kan extensions “weak Kan extensions,” reserves “Kan extensions” for what we call pointwise Kan extensions, and even says:

Our present choice of nomenclature is based on our failure to find a single instance where a weak Kan extension plays any mathematical role whatsoever.

I leave it to each of you to test this rule of thumb in your future encounters with Kan extensions.

In Theorem 6.1, the more subsets of R have suprema, the more likely a left Kan extension is to exist. The extreme case is a complete lattice.

Definition 6.7 (Complete lattice). A poset P is a *complete lattice* if every subset of P has a supremum.¹⁰

Corollary 6.8 (Existence of Kan extensions). If R is a complete lattice, then every pair of order-preserving maps $f : P \rightarrow R$, $g : P \rightarrow Q$ has a pointwise left Kan extension $\text{Lan}_g f$.

Among our examples, Ω , $\mathcal{P}(X)$, $\mathcal{O}(X)$, $[0, \infty]$, and the various families of closed subsets from the exercises (closed sets of a topological space, subalgebras, convex subsets, and so on) are complete lattices. In $\mathbb{R}$, the only subsets without suprema are the empty set¹¹ and those unbounded above. A set with the trivial order, or the poset $\mathbb{Q}$, has many subsets without suprema,¹² which is why Kan extensions were harder to come by there.

Here is the converse of Corollary 6.8, as an exercise.

Exercise 6.9 (The converse existence theorem). Let R be a poset. Show that the following three conditions are equivalent. (Hint: Example 4.9.)

1. R is a complete lattice.
2. Every pair of order-preserving maps $f : P \rightarrow R$, $g : P \rightarrow Q$ has a pointwise left Kan extension $\text{Lan}_g f$.
3. Every pair of order-preserving maps $f : P \rightarrow R$, $g : P \rightarrow Q$ has a left Kan extension $\text{Lan}_g f$.

7 Some interesting properties of the formula

Using the formula from Section 6, we will explore interesting properties of Kan extensions, including a sufficient condition for being an actual extension. This section is more theoretical than the others, so feel free to do only the exercises that interest you.

Another aim of this section is to help with learning Kan extensions in category theory. The properties below are poset versions of general theorems about Kan extensions in category theory. The proofs, presented as exercises, work just as in the general case. To stay within the aims of this text, though, I will only mention the categorical background, not explain it—even though it is interesting, powerful, and very important for an intuitive understanding. If you plan to learn the corresponding categorical theorems, read this for a first encounter with their ideas in a simple setting. If you already know them, read it for a better understanding through simple examples.

Definition 7.1 (Order embedding). An order-preserving map $f : P \rightarrow Q$ is an *embedding* if, for all $p, p' \in P$,

$$p \leq p' \iff f(p) \leq f(p')$$

holds.

Its categorical counterpart is a fully faithful functor: Definition 1.5.7 of [1]. Every order-preserving map we have called an embedding so far is indeed an embedding in this sense.

Exercise 7.2 (A sufficient condition for an actual extension). Suppose that $g : P \rightarrow Q$ is an order embedding. For any order-preserving map $f : P \rightarrow R$, show that its pointwise left Kan extension along g satisfies $f = (\text{Lan}_g f) \circ g$

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g & \nearrow \text{Lan}_g f \\ & Q & \end{array} \quad \begin{array}{c} \\ \parallel \\ \end{array}$$

as displayed above. **Pointwise Kan extensions along embeddings are actual extensions.** The corresponding categorical theorem is Corollary 6.3.9 of [1].

¹⁰Equivalently, every subset of P has an infimum.

¹¹We regard the supremum of the empty set as the least element.

¹²The one-element poset is an exception: it is a complete lattice.

In Section 8, we will see why the pointwise assumption is needed.

Exercise 7.3 (Another definition of pointwiseness). For an element r of a poset R , consider the order-preserving map $r \leq - : R \rightarrow \Omega$. Let $\text{Ran}_g f$ be the right Kan extension of $f : P \rightarrow R$ along $g : P \rightarrow Q$. Show that this Kan extension is pointwise if and only if, for every $r \in R$,

$$\begin{array}{ccc} P & \xrightarrow{f} & R \xrightarrow{r \leq -} \Omega \\ & \searrow g \quad \vee \mid \nearrow \text{Ran}_g f & \\ & Q & \end{array} = \begin{array}{ccc} P & \xrightarrow{f} & R \xrightarrow{r \leq -} \Omega \\ & \searrow g \quad \vee \mid \nearrow \text{Ran}_g f & \\ & Q & \end{array}$$

is again a right Kan extension. The categorical counterpart is Theorem 6.3.7 of [1].

Using this alternative definition explicitly, let us give a sufficient condition for pointwiseness.

Exercise 7.4 (A least element ensures pointwiseness). Consider a right Kan extension

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g \quad \vee \mid \nearrow \text{Ran}_g f & \\ & Q & \end{array}$$

We want to show that **if R has a least element 0_R , then $\text{Ran}_g f$ is pointwise**. Let us use Exercise 7.3. Take any $r \in R$.

$$\begin{array}{ccc} P & \xrightarrow{f} & R \xrightarrow{r \leq -} \Omega \\ & \searrow g \quad \vee \mid \nearrow \text{Ran}_g f & \\ & Q & \end{array} = \begin{array}{ccc} P & \xrightarrow{f} & R \xrightarrow{r \leq -} \Omega \\ & \searrow g \quad \vee \mid \nearrow \text{Ran}_g f & \\ & Q & \end{array}$$

It is enough to show that this is a right Kan extension. To avoid confusing notation, write $s_r : R \rightarrow \Omega$ for $r \leq -$. As an auxiliary map in the other direction, define $t_r : \Omega \rightarrow R$ by

$$\begin{aligned} t_r(\text{true}) &= r \\ t_r(\text{false}) &= 0_R. \end{aligned}$$

1. Show that $t_r \circ s_r \leq \text{id}_R$ and $\text{id}_\Omega \leq s_r \circ t_r$.¹³

$$\begin{array}{ccc} R & \xrightarrow{\text{id}_R} & R \\ & \searrow s_r \quad \vee \mid \nearrow t_r & \\ & \Omega & \end{array} \quad \begin{array}{ccc} & R & \\ t_r \nearrow & \vee \mid & \searrow s_r \\ \Omega & \xrightarrow{\text{id}_\Omega} & \Omega \end{array}$$

2. Using the first part of 1, show that any order-preserving map $h : Q \rightarrow \Omega$ with $hg \leq s_r \circ f$ satisfies $t_r hg \leq f$.

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ & \searrow g \quad \vee \mid \nearrow s_r & \\ & Q & \xrightarrow{h} \Omega \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} P & \xrightarrow{f} & R & \xrightarrow{\text{id}_R} & R \\ & \searrow g \quad \vee \mid \nearrow s_r & & \vee \mid \nearrow t_r & \\ & Q & \xrightarrow{h} & \Omega & \end{array}$$

¹³This is called a Galois connection in order theory and an adjunction in category theory.

3. Use 2 and the definition of $\text{Ran}_g f$ to show that $t_r h \leq \text{Ran}_g f$.

$$\begin{array}{ccc} Q & \xrightarrow{\text{Ran}_g f} & R \\ & \searrow h \quad \swarrow t_r & \\ & \Omega & \end{array} \quad \begin{array}{c} \vee \\ | \end{array}$$

4. Use the second part of 1 and part 3 to show that $h \leq s_r \text{Ran}_g f$.

$$\begin{array}{ccccc} Q & \xrightarrow{\text{Ran}_g f} & R & & \\ & \searrow h \quad \swarrow t_r & & \searrow s_r & \\ & \Omega & \xrightarrow{\text{id}_\Omega} & \Omega & \end{array} \quad \begin{array}{c} \vee \\ | \end{array}$$

5. Conclude from the above and Exercise 7.3 that $\text{Ran}_g f$ is a pointwise right Kan extension.

In [1], this exercise corresponds to using Lemma 6.3.2 to solve Exercise 6.3.i.

To prepare for the next two exercises, let us define one more order-preserving map.

Definition 7.5 ($y_P : P \rightarrow \mathcal{D}(P)$). For a poset P , define a poset $\mathcal{D}(P)$ and an order-preserving map $y_P : P \rightarrow \mathcal{D}(P)$ as follows.

- A subset $S \subset P$ is *downward closed* if, whenever $p \in S$, every element of P below p also belongs to S .
- Write $\mathcal{D}(P)$ for the poset of downward-closed subsets of P , ordered by inclusion.
- Define the order-preserving map $y_P : P \rightarrow \mathcal{D}(P)$ by

$$y_P(p) := \{p' \in P \mid p' \leq p\}$$

The categorical counterparts, including the fact that y_P is an embedding, are the presheaf category and the Yoneda embedding: Corollary 2.2.8 of [1].

Exercise 7.6 (The universal property of a power set). Regard a set X as a poset with the trivial order. The (order-preserving) map $y_X : X \rightarrow \mathcal{D}(X) = \mathcal{P}(X)$ is given by $y_X(x) = \{x\}$. The power set $\mathcal{P}(X)$ is a complete lattice (see Remark 6.7). Let L be a complete lattice. For a (necessarily order-preserving) map $f : X \rightarrow L$, define $\hat{f} : \mathcal{P}(X) \rightarrow L$ as the left Kan extension

$$\begin{array}{ccc} X & \xrightarrow{f} & L \\ & \searrow y_X \quad \swarrow \hat{f} & \\ & \mathcal{P}(X) & \end{array} \quad \begin{array}{c} \wedge \\ | \end{array}$$

whose existence is guaranteed by Theorem 6.1.

1. Show that $\hat{f}$ preserves all suprema.¹⁴
2. Show that $f \mapsto \hat{f}$ gives a bijection between (order-preserving) maps $X \rightarrow L$ and supremum-preserving maps $\mathcal{P}(X) \rightarrow L$. Thus **the power set is the universal completion of a set**.¹⁵

¹⁴Preserving suprema means that, for every subset, the image of its supremum is the supremum of its image.

¹⁵In categorical terms, we should really say cocompletion.

3. Define $\check{f} : L \rightarrow \mathcal{P}(X)$ by $\check{f}(l) := \{x \in X \mid f(x) \leq l\}$. Show that, for every $S \in \mathcal{P}(X)$ and $l \in L$,

$$S \subset \check{f}(l) \iff \hat{f}(S) \leq l$$

holds.¹⁶

4. Generalize from the trivially ordered set X to an arbitrary poset P , and conclude that $\mathcal{D}(P)$ is the **universal completion** of P .

The categorical counterpart is nerve and realization,¹⁷ one of the most interesting theorems about Kan extensions. See <https://ncatlab.org/nlab/show/nerve+and+realization> or Remark 6.5.9 of [1].

Exercise 7.7 (Density). An order-preserving map $f : Q \rightarrow R$ is *dense* if id_R is the **pointwise** left Kan extension of f along itself.

$$\begin{array}{ccc} Q & \xrightarrow{f} & R \\ & \searrow f \quad \wedge \downarrow \quad \nearrow \text{id}_R & \\ & R & \end{array}$$

Informally, every element of R can be reconstructed, using the formula, solely from its relations to the elements of Q .

- For an order-preserving map $f : Q \rightarrow R$, define $\check{f} : R \rightarrow \mathcal{D}(Q)$ by $\check{f}(r) := \{q \in Q \mid f(q) \leq r\}$.¹⁸ Show that f is dense if and only if $\check{f}$ is an embedding: as displayed above. (Elements of R can be described as subsets of Q .)
- Show that $y_P : P \rightarrow \mathcal{D}(P)$ is dense for every poset P .

The categorical counterpart is Theorem 6.5.8 of [1], or Chapter X, Section 6 of [2].

For what happens without pointwiseness, see Section 8. In the density exercise, pointwiseness may have seemed to appear out of nowhere in the definition; solving the exercise alone may not have made its motivation clear. So here is a familiar example of density.

Example 7.8 (Density in the reals). For a subset $S \subset \mathbb{R}$, the inclusion $\iota : S \rightarrow \mathbb{R}$ is dense in this sense exactly when S is topologically dense in $\mathbb{R}$: between any two distinct reals $r < r'$ lies an $s \in S$ with $r < s < r'$. In particular, $\iota : \mathbb{Q} \rightarrow \mathbb{R}$ is dense. By part 1 of Exercise 7.7, $\check{\iota} : \mathbb{R} \rightarrow \mathcal{D}(\mathbb{Q})$ is an embedding, so real numbers can be described as downward-closed subsets of $\mathbb{Q}$. These are the familiar **Dedekind cuts**

8 Unexpected toys

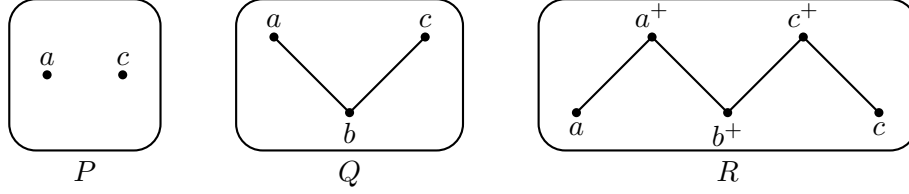
Here are four slightly unusual Kan extensions. The first two are not pointwise, and provide counterexamples to the two exercises in Section 7 if we drop their pointwise assumptions. The other two describe Cantor's devil's staircase as a Kan extension and give a **proof** of the Cayley–Hamilton theorem using Kan extensions.

Example 8.1 (A nonpointwise Kan extension, I). Consider these three posets. Give $P = \{a, c\}$ the trivial order. Give $Q = \{a, b, c\}$ the order $a \geq b \leq c$. Give $R = \{a, a^+, b^+, c^+, c\}$ the order $a \leq a^+ \geq b^+ \leq c^+ \geq c$.

¹⁶This is a Galois connection in order theory, or an adjunction in category theory.

¹⁷Apparently also called Kan's adjunction or the universal adjunction.

¹⁸This is the same map as in Exercise 7.6. Categorically, it is the restricted Yoneda embedding.



Define $f : P \rightarrow R$ and $g : P \rightarrow Q$ by sending a to a and c to c . Define $(-)^+ : Q \rightarrow R$ by sending a, b, c to a^+, b^+, c^+ , respectively. These maps are order-preserving.

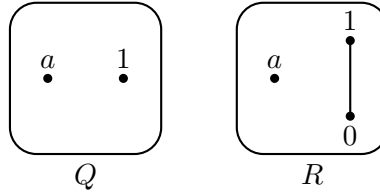
Then $f \leq (-)^+ \circ g$.

$$\begin{array}{ccc}
 P & \xrightarrow{f} & R \\
 & \searrow g \quad \wedge \mid \quad \nearrow (-)^+ & \\
 & Q &
 \end{array}$$

Moreover, $(-)^+$ is the only order-preserving map $h : Q \rightarrow R$ satisfying $f \leq h \circ g$. Thus $(-)^+$ is the left Kan extension of f along g . But **this left Kan extension is not pointwise**. Also, although g is an embedding, $(-)^+$ is not an actual extension of f . So **the pointwise assumption in Exercise 7.2 is necessary**.

The next example was shown to me by @LiEat_D. I have rewritten it, so any errors are my responsibility.

Example 8.2 (A nonpointwise Kan extension, II). Consider two posets. Give $Q = \{a, 1\}$ the trivial order. Give $R = \{a, 0, 1\}$ the order $0 \leq 1$.



Define $f : Q \rightarrow R$ by sending a to a and 1 to 1 . Then id_R is the left Kan extension of f along f .

$$\begin{array}{ccc}
 Q & \xrightarrow{f} & R \\
 & \searrow f \quad \wedge \mid \quad \nearrow \text{id}_R & \\
 & R &
 \end{array}$$

But looking at the image of $0 \in R$ shows that this Kan extension is not pointwise. Thus **dropping pointwiseness from the definition of density in Exercise 7.7 gives a different notion**, and of course part 1 of Exercise 7.7 no longer holds.

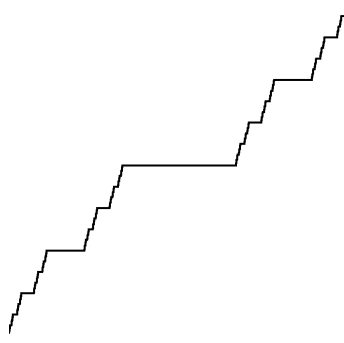
Example 8.3 (Cantor's devil's staircase). Let us see why Cantor's devil's staircase is the right Kan extension of binary evaluation along ternary evaluation.

Consider the poset of infinite strings in two symbols $+, -$, ordered lexicographically with $+\geq -$.¹⁹ Write this poset as $\text{Sq} := \{+, -\}^{\{1,2,3,\dots\}}$. For a positive integer $m \geq 1$, define the order-preserving base- m evaluation map $\text{ev}_m : \text{Sq} \rightarrow [0, 1]$ as follows. Replace $-$ by 0 and $+$ by $m - 1$, and read the result as a base- m fractional expansion. In other words, put

$$\text{ev}_m((a_n)_{n=1}^\infty) := \sum_{n \geq 1, a_n = +} \frac{m-1}{m^n}$$

¹⁹That is, $(a_n)_{n=1}^\infty \leq (b_n)_{n=1}^\infty$ if the strings are equal, or if their first differing symbols satisfy $a_{n_0} = -$ and $b_{n_0} = +$.

Figure 1: Cantor’s devil’s staircase



For example,

$$\begin{aligned} \text{ev}_{10}(+++++\dots) &= 0.999999\dots = 1 \\ \text{ev}_3(+ - + - + - \dots) &= (0.202020\dots)_3 = 0.75 \end{aligned}$$

Cantor’s devil’s staircase $c : [0, 1] \rightarrow [0, 1]$ is the right Kan extension of ev_2 along ev_3 .

$$\begin{array}{ccc} \text{Sq} & \xrightarrow{\text{ev}_2} & [0, 1] \\ & \searrow \text{ev}_3 \quad \swarrow c & \\ & [0, 1] & \end{array} \quad \text{with } \text{ev}_2 = c \circ \text{ev}_3$$

Example 8.4 (The Cayley–Hamilton theorem). Call a pair (V, f) consisting of a finite-dimensional vector space V over $\mathbb{C}^{20}$ and a linear endomorphism $f : V \rightarrow V$ a finite-dimensional linear discrete dynamical system, or FLDD for short. The theory of square matrices in linear algebra can be viewed as the theory of FLDDs.

Recall that the minimal polynomial $\varphi_{(V,f)}$ of an FLDD is the monic polynomial²¹ $\varphi(X) \in \mathbb{C}[X]$ of least degree satisfying $\varphi(f) = 0$. This definition has the form “the least thing satisfying a condition,” so it is a kind of universal property. Let us express it as a supremum and see that it is a (pointwise) left Kan extension.

The linear map $\varphi(f)$ is zero precisely when it sends every element to zero. For each $v \in V$, let $\varphi_v(X)$ be the monic polynomial of least degree with $\varphi_v(f)(v) = 0$. Then the minimal polynomial $\varphi(X)$ is the least common multiple of $\{\varphi_v \mid v \in V\}$. Least common multiples are suprema in the poset $\text{MPoly}_{\mathbb{C}}$ of monic polynomials in $\mathbb{C}[X]$, ordered by divisibility.²² Thus we can write

$$\varphi(X) = \sup_{v \in V} \varphi_v(X)$$

Let us see this as a pointwise left Kan extension. Define LDyn to be the poset of isomorphism classes²³ of FLDDs, with $(W, g) \leq (V, f)$ when there is an injective linear map $(W, g) \rightarrow (V, f)$ commuting with the endomorphisms. and let LDyn^{cyc} be the subposet of cyclic FLDDs.²⁴ Since the minimal polynomial of a subsystem divides that of the whole system, taking minimal polynomials gives an order-preserving map $\varphi_- : \text{LDyn} \rightarrow \text{MPoly}_{\mathbb{C}}$. We use the same notation for

²⁰Or any field you like.

²¹A polynomial whose leading coefficient is 1.

²²If you know ring theory, I recommend using instead the opposite of the poset of ideals of $\mathbb{C}[X]$ under inclusion. Since $\mathbb{C}[X]$ is a PID, this gives an order-reversing isomorphism, depending on whether we include 0 among the monic polynomials. Working with ideals simplifies the discussion (no need to keep saying “monic”), allows generalizations in several directions, and lets us immediately use general facts about posets of subalgebras, such as completeness.

²³Isomorphism means the existence of a linear isomorphism commuting with the endomorphisms.

²⁴An FLDD (V, f) is cyclic if there is a $v \in V$ such that $\{f^n(v) \mid n \in \mathbb{N}\}$ spans V . Equivalently, some $v \in V$ has the property that the smallest linear dynamical subsystem containing it is all of V .

its restriction $\varphi_- : \text{LDyn}^{\text{cyc}} \rightarrow \text{MPoly}_{\mathbb{C}}$. Replacing individual elements by the cyclic subsystems they generate, our discussion shows that

$$\begin{array}{ccc} \text{LDyn}^{\text{cyc}} & \xrightarrow{\varphi_-} & \text{MPoly}_{\mathbb{C}} \\ & \searrow \iota \quad \wedge \mid \quad \nearrow \varphi_- & \\ & \text{LDyn} & \end{array}$$

is a pointwise left Kan extension. Thus the minimal polynomial of (V, f) is the natural extension, to all of V , of the polynomials annihilating individual elements of V .

The fact that minimal polynomials form a Kan extension actually implies the Cayley–Hamilton theorem. For a cyclic FLDD, the minimal and characteristic polynomials coincide.²⁵ So, if $\Phi_- : \text{LDyn} \rightarrow \text{MPoly}_{\mathbb{C}}$ is the order-preserving map taking characteristic polynomials,²⁶ we have $\varphi_- = \Phi_- \circ \iota : \text{LDyn}^{\text{cyc}} \rightarrow \text{MPoly}_{\mathbb{C}}$.

$$\begin{array}{ccc} \text{LDyn}^{\text{cyc}} & \xrightarrow{\varphi_- = \Phi_-} & \text{MPoly}_{\mathbb{C}} \\ & \searrow \iota \quad \parallel \quad \nearrow \Phi_- & \\ & \text{LDyn} & \end{array}$$

By the defining universal property of the left Kan extension,

$$\begin{array}{ccc} \text{LDyn}^{\text{cyc}} & \xrightarrow{\varphi_- = \Phi_-} & \text{MPoly}_{\mathbb{C}} \\ & \searrow \iota \quad \wedge \mid \quad \nearrow \Phi_- & \\ & \text{LDyn} & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \text{LDyn}^{\text{cyc}} & \xrightarrow{\varphi_- = \Phi_-} & \text{MPoly}_{\mathbb{C}} \\ & \searrow \iota \quad \wedge \mid \quad \nearrow \varphi_- & \\ & \text{LDyn} & \end{array}$$

we obtain $\varphi_- \leq \Phi_- : \text{LDyn} \rightarrow \text{MPoly}_{\mathbb{C}}$. In other words, for every FLDD, **the minimal polynomial divides the characteristic polynomial**. This is the Cayley–Hamilton theorem.

This proof becomes a little cleaner in the wider setting of category theory. The proof of the Cayley–Hamilton theorem in [7] is essentially this argument written out explicitly.

Here is an exercise that is half a joke and half serious.

Exercise 8.5. Look for Kan extensions between posets in mathematics you have learned recently. Whenever you find an interesting one, please tell me (Twitter: @hora_algebra).

9 A glimpse of category theory: Kan extensions can do even more!

So far, we have explained Kan extensions within the world of posets. Usually, they are discussed in category theory. If you would like to learn about that setting, try [1] or [2].

A poset describes order relations between elements. A category generalizes this by allowing richer relationships (usually called morphisms) between its elements (usually called objects). One source of category theory’s power is that mathematical objects and the relationships between them can form a category. The counterpart of an order-preserving map—a functor—can then be viewed as a mathematical construction. From this perspective, **Kan extensions become a**

²⁵For a generator v , look at the matrix in the basis $\{f^n(v) \mid 0 \leq n < \dim V\}$.

²⁶Order preservation is easy to check. For a subsystem W , write the matrix in a basis extending a basis of W .

theory of extending mathematical constructions, rather than merely a theory of extending order-preserving maps. This is one of the essential reasons they matter.

Basic category theory also includes the Yoneda lemma, Yoneda embeddings, adjunctions, limits, monads, and more. Kan extensions give an intuitive, unified perspective on these ideas. For example, “limits and colimits are the two natural ways to obtain an object from a functor.”²⁷ In this sense, **Kan extensions provide a unified theory of the basic concepts of category theory**. This is the viewpoint behind the famous phrase “All Concepts Are Kan Extensions.” To quote [2]:

The notion of Kan extensions subsumes all the other fundamental concepts of category theory.

Rather than finish by listing all the benefits of learning category theory, I want to highlight one role you may already have felt in this text: **category theory as a guide for the rest of us**. Category theory tries to offer natural, intuitive perspectives on mathematics. Even if you are not a genius who can find the right path unaided, the category theory organized by earlier geniuses may point you in the “right” direction. Suppose you want to extend a measure as far as possible. You need not think of starting with “every subset of every null set...”: knowing Kan extensions between posets, you can simply take your wish to extend as far as possible to the theory of Kan extensions. Once you learn Kan extensions in category theory, you can leave posets behind. Whether you want to extend a representation of a subgroup to the whole group, extend a functor to a category with inverses to quasi-isomorphisms adjoined, extend the standard simplices indexed by natural numbers to topological realizations of simplicial sets, or extend the way we view open sets as bundles to a way of viewing presheaves as bundles, Kan extensions can point you in the “right” direction.

References

- [1] Riehl, Emily. Category theory in context. Courier Dover Publications, 2017.
- [2] Mac Lane, Saunders. Categories for the working mathematician. Vol. 5. Springer Science & Business Media, 2013.
- [3] MacLane, Saunders, and Ieke Moerdijk. Sheaves in geometry and logic: A first introduction to topos theory. Springer Science & Business Media, 2012.
- [4] Awodey, Steve. Category theory. Oxford university press, 2010.
- [5] Kelly, G. M.. BASIC CONCEPTS OF ENRICHED CATEGORY THEORY. 2005.
- [6] hora-algebra, *Set Operations as Toy Examples in Category Theory* (in Japanese). <https://hora-algebra.github.io/ToyExample1.pdf>
- [7] Takeshi Saito. *The World of Linear Algebra: An Introduction to Abstract Mathematics* (in Japanese). University of Tokyo Press, 2007.

²⁷The poset counterpart is Exercise 4.9.