Combinatorial games as recursive coalgebras

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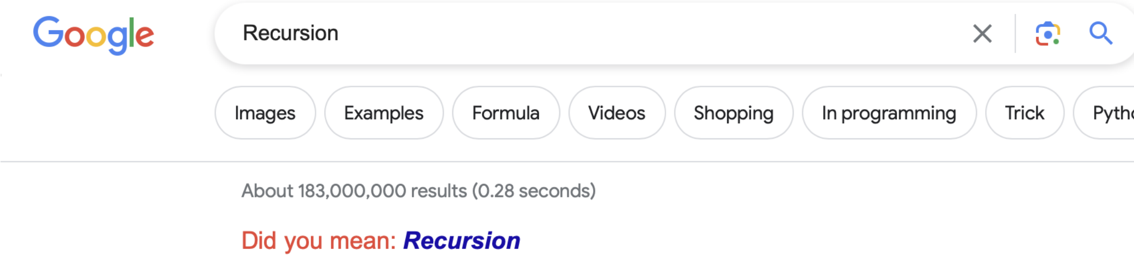
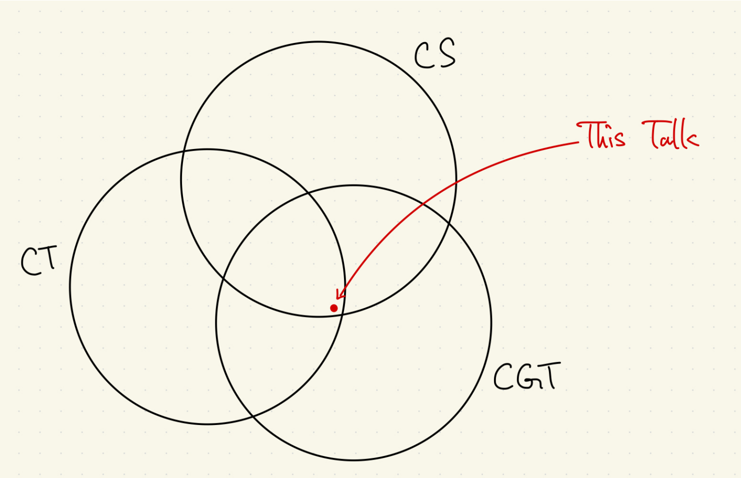
Self Introduction

Ryuya Hora

Recently I've been interested in toposes, ∞-categories, and **coalgebras**.



CT, CS, CGT



This talk in one slide

- Basics of CGT: The *generalized Bouton's theorem* allows us to decompose a game into smaller parts
- We prove **Games =** ***Recursive*** $\mathcal{P}_{\mathrm{fin}}$***-coalgebras***
- As an application, we obtain a *generalized generalized Bouton's theorem*, which may be a very powerful tool in CGT.

CGT	CT
games	$=$ recursive $\mathcal{P}_{\mathrm{fin}}$-coalgebras
Winning/Losing state	$\in$ $\mathcal{P}_{\mathrm{fin}}$-algebra
Grundy number	$\in$ $\mathcal{P}_{\mathrm{fin}}$-algebra
Conway addition	$\in$ Monoidal structure
nim-sum (!?)	$\in$ *Bouton monoid*
gen. Bouton's theorem	$\in$ gen. gen. Bouton's theorem!

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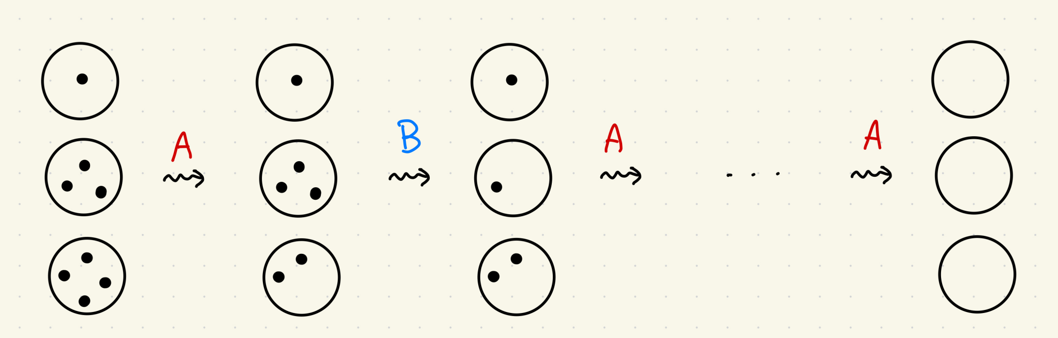
Rule of n-heap Nim

- n heaps of stones are given.
- Two players take turns choosing one heap and removing at least one stone from that heap.
- The player who is unable to take a stone loses.



Bouton's theorem

Definition (Nim-sum)

The *Nim-sum* $\oplus$ is "bit-wise xor", i.e., an abelian group structure on $\mathbb{N}$, induced by the binary expansion $\mathbb{N} \xrightarrow{\cong} \bigoplus_{k=0}^{\infty} \mathbb{Z}/2\mathbb{Z}$.

Example

$5 \oplus 7 = (101)_2 \oplus (111)_2 = (010)_2 = 2$

Theorem ([Bouton, Ann. of Math., 1902])

A state of n-heap nim $(a_1, \ldots, a_n)$ *is winning state if and only if* $a_1 \oplus \ldots \oplus a_n = 0$.

Example

$(1, 2, 3), (0, 1, 1), (2, 2, 0)$ are winning states of the 3-heap nim.

Motivation

Motivation

Where does nim-sum come from?

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Where does nim-sum come from?

$\rightarrow$Let's abstract the essence of Bouton's theorem!

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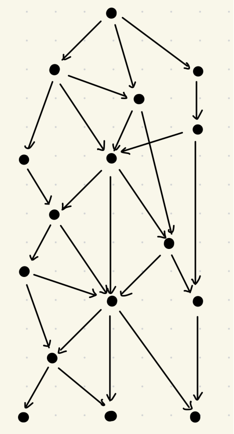
Definition of games

Definition (Game)

A *game* $\mathbb{X} = (X, \to)$ is a pair of a (possibly infinite) set X and a binary relation $\to \subset X \times X$ that satisfies the following two finiteness conditions

1. (finite options) $\#\{x' \in X \mid x \to x'\}$ is finite, for any $x \in X$.
2. (finite time) There is no infinite path. $x_0 \to x_1 \to x_2 \to \ldots$



Example (Nim_n: *n*-heap nim)

The game $\mathrm{Nim}_n = (\mathbb{N}^n, \to)$ is defined by

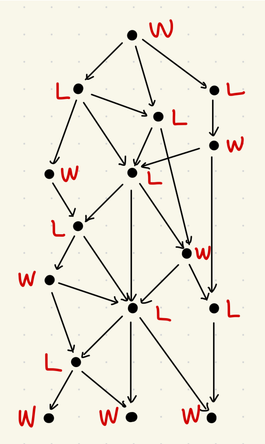
$$(a_i)_{1 \le i \le n} \to (b_i)_{1 \le i \le n} \iff \exists i (a_i > b_i \wedge a_j = b_j (j \neq i))$$

Outcome: Winning/Losing state

Definition (Outcome)

For a game $\mathbb{X} = (X, \to)$ and a state $x \in X$, its *outcome* $\mathcal{O}_{\mathbb{X}}(x) \in \{W, L\}$ is recursively defined by

$$\mathcal{O}_{\mathbb{X}}(x) := \begin{cases} W & (x \to \forall x', \mathcal{O}_{\mathbb{X}}(x') = L) \\ L & (x \to \exists x', \mathcal{O}_{\mathbb{X}}(x') = W) \end{cases}$$



Grundy number

Definition (mex)

The *mex* of a finite subset $S \subset \mathbb{N}$ is $\min(\mathbb{N} \setminus S)$.

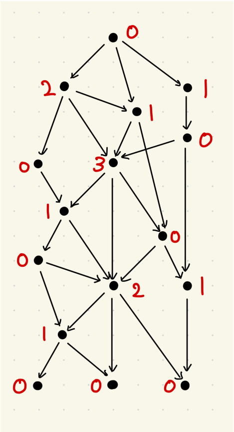
Definition (Grundy number)

For a game $\mathbb{X} = (X, \to)$ and a state $x \in X$, its *Grundy number* $\mathcal{G}_{\mathbb{X}}(x)$ is recursively defined by

$$\mathcal{G}_{\mathbb{X}}(x) := \mathrm{mex}(\{\mathcal{G}_{\mathbb{X}}(x') \mid x \to x'\})$$

Proposition

For a game $\mathbb{X} = (X, \to)$, a state x is a winning state if and only if $\mathcal{G}_{\mathbb{X}}(x) = 0$.



Conway addition of games

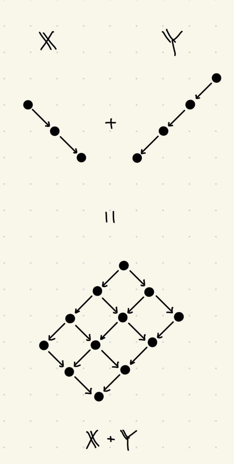
Definition (Conway addition)

The Conway addition of two games, $\mathbb{X} = (X, \rightarrow_X)$ and $\mathbb{Y} = (Y, \rightarrow_Y)$, is the game $\mathbb{X} + \mathbb{Y} = (X \times Y, \rightarrow_+)$, where

- the underlying set is the cartesian product $X \times Y$, and
- the relation $\rightarrow_+$ is defined by $(x, y) \rightarrow_+ (x', y') \iff (x \rightarrow_X x' \wedge y = y') \vee (x = x' \wedge y \rightarrow_Y y')$

Example

$\mathrm{Nim}_n = \mathrm{Nim}_1 + \cdots + \mathrm{Nim}_1$



Generalized Bouton's theorem

Theorem (Generalized Bouton's theorem [see CGT, Siegel])

For two games $\mathbb{X} = (X, \rightarrow_X)$ *and* $\mathbb{Y} = (Y, \rightarrow_Y)$, *we have*

$$\mathcal{G}_{\mathbb{X}+\mathbb{Y}}(x, y) = \mathcal{G}_{\mathbb{X}}(x) \oplus \mathcal{G}_{\mathbb{Y}}(y).$$

Example (Original Bouton's theorem)

1. $\mathcal{G}_{\mathrm{Nim}_1}(a) = a$
2. $\mathcal{G}_{\mathrm{Nim}_n}((a_i)_{1 \le i \le n}) = \mathcal{G}_{\mathrm{Nim}_1}(a_1) \oplus \ldots \oplus \mathcal{G}_{\mathrm{Nim}_1}(a_n) = a_1 \oplus \ldots \oplus a_n$
3. $(a_i)_{1 \le i \le n}$ is a winning state $\iff a_1 \oplus \ldots \oplus a_n = 0$

Generalized Bouton's theorem allows us to decompose a game into smaller parts!

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3. $(a_i)_{1 \le i \le n}$ is a winning state $\iff a_1 \oplus \ldots \oplus a_n = 0$

Generalized Bouton's theorem allows us to decompose a game into smaller parts!
(And the game Nim_n is not the essense of the nim-sum!)

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What is a morphism of games?: Historical Remarks

Historically, morphisms of games have been (relative) *strategies*:

- A. Joyal, Remarques sur la théorie des jeux à deux personnes, 1977
- M. Hyland, Game semantics, 1997
- J.Baez, Classical vs Quantum Computation (Week 3) (The n-Category Café), 2006.

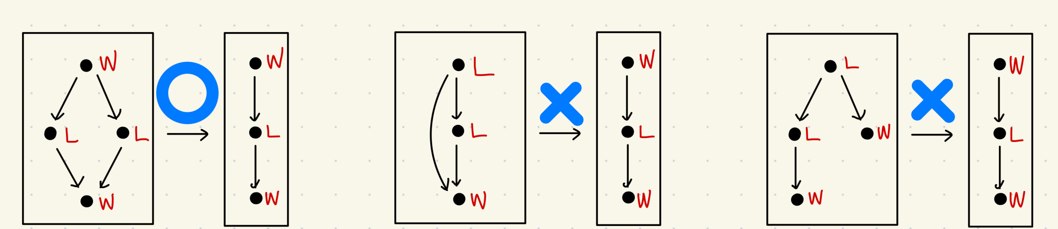
However, we will adopt a more classical category-theoretic approach, which is to define a morphism as a "structure-preserving function".

What is a morphism of games?: Our definition

Definition (Game morphisms)

A *game morphism* $f : (X, \to_X) \to (Y, \to_Y)$ is a function $f : X \to Y$ that satisfies the following two conditions:

1. (Graph map) if $x \to_X x'$ then $f(x) \to_Y f(x')$.
2. (Lifting property) if $f(x) \to_X y$, then there exists $x \to_X x'$ such that $f(x') = y$.

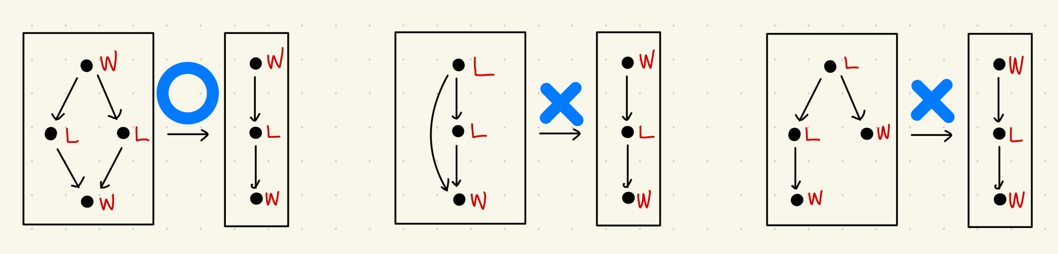


We can prove that game morphisms preserve "game-theoretic data":

Proposition

Game morphisms preserve

- *Winning/Losing states,*
- *Grundy numbers,*
- *(and any "recursive data.")*



Categorical structure of games

The category of games **Games** has good categorical properties, including:

Proposition (Games is LFP.)

The category of games **Games** *is locally finitely presentable. In particular, it is complete and cocomplete.*

Colimits are created by $U\colon \mathbf{Games} \to \mathrm{Set}$, but limits are non-trivial!

Example (The terminal game: $T = (V_\omega, \ni)$)

The terminal game $T = (\mathbb{N}, \to_{\mathrm{bin}})$ is the *binary nim*. For $n, m \in \mathbb{N}$, $n \to_{\mathrm{bin}} m$, if m appears in the binary expasion of n. For example,

$$10000 = 2^4 + 2^8 + 2^9 + 2^{10} + 2^{13} \to_{\mathrm{bin}} 4, 8, 9, 10, 13.$$

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Algebra/Coalgebra of an endofunctor

Definition (T-Algebras and T-Coalgebras)

For a category $\mathcal{C}$ and an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$,

- A T-algebra is a pair (A, α) of an object A of $\mathcal{C}$ and a morphism $\alpha\colon TA \to A$.
- A T-coalgebra is a pair (X, θ) of an object X of $\mathcal{C}$ and a morphism $\theta\colon X \to TX$.

Example

We will consider the case where $\mathcal{C} = \mathrm{Set}$ and $T = \mathcal{P}_{\mathrm{fin}}\colon \mathrm{Set} \to \mathrm{Set}$.

$$\mathcal{P}_{\mathrm{fin}}(X) = \{S \subset X \mid \#S < \infty\}$$

Coalgebra-Algebra morphism and Recursive coalgebra

Definition (Coalgebra-algebra morphism)

A *coalgebra-algebra* morphism from a T-coalgebra (X, θ) to a T-algebra (A, α) is a morphism $f : X \to A$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta"] & A \\ TX \arrow[r, "Tf"] & TA \arrow[u, "\alpha"'] \end{tikzcd}$$

Definition (Recursive coalgebra)

A T-coalgebra (X, θ) is *recursive* if for any T-algebra (A, α), there uniquely exists a coalgebra-algebra morphism $(X, \theta) \to (A, \alpha)$.

Games = Recursive $\mathcal{P}_{\text{fin}}$-coalgebras

Theorem (Games as Recursive coalgebras)

The category of games **Games** *is equivalent to the category of recursive $\mathcal{P}_{\text{fin}}$-coalgebras.*

$$
\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[d, mapsto] & & & g_X(x) \arrow[r, phantom, "="] & \operatorname{mex}\{g_X(x') \mid x \to x'\} \\
X \arrow[r, "g_X"] \arrow[d, "\theta"'] \arrow[rd, phantom, "\circlearrowright"] & \mathbb{N} & & & \\
\mathcal{P}_{\text{fin}}(X) \arrow[r, "\mathcal{P}_{\text{fin}}(g_X)"'] & \mathcal{P}_{\text{fin}}(\mathbb{N}) \arrow[u, "\operatorname{mex}"'] & & & \\
\theta(x) := \{x' \mid x \to x'\} \arrow[rrrr, mapsto] & & & & \{g_X(x') \mid x \to x'\} \arrow[uuu, mapsto]
\end{tikzcd}
$$

Recursive values $=$ $\mathcal{P}_{\mathrm{fin}}$-algebras

For a $\mathcal{P}_{\mathrm{fin}}$-algebra $\mathbb{A} = (A, \alpha)$ and a game $\mathbb{X} = (X, \to)$, there exists a unique function $\mathrm{h}_{\mathbb{A},\mathbb{X}} \colon X \to A$ such that

$$\mathrm{h}_{\mathbb{A},\mathbb{X}}(x) = \alpha(\{\mathrm{h}_{\mathbb{A},\mathbb{X}}(x') \mid x \to x'\})$$

Example

"Recursive value"	$\leftrightsquigarrow$	$\mathcal{P}_{\mathrm{fin}}$-algebra
W/L states	$\leftrightsquigarrow$	$\mathcal{P}_{\mathrm{fin}}(\{W, L\}) \longrightarrow \{W, L\}$
Grundy number	$\leftrightsquigarrow$	$\mathcal{P}_{\mathrm{fin}}(\mathbb{N}) \xrightarrow{\mathrm{mex}} \mathbb{N}$

Corollary

Game morphisms preserve all "recursive values."

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Let's return to our original motivation!

Motivation

Where does nim-sum come from?

In this section,

- let $*$ be a monoidal structure on **Games** such that the forgetful functor

$$U\colon \textbf{Games} \to \mathrm{Set}$$

  is lax monoidal, and
- let $\mathbb{A} = (A, \alpha)$ be a $\mathcal{P}_{\mathrm{fin}}$-algebra.

Example

The prototypical example is the Conway addition $* = +$ and the W/L algebra $\mathbb{A} = (\{W, L\}, \alpha)$.

Bouton monoid

Definition (Bouton monoid)

Then we can construct the *Bouton monoid* $\mathcal{B}_{*,\mathbb{A}}$ by the following procedure:

1. The terminal game T has the unique monoid structure w.r.t. $(\mathbf{Games}, *)$.
2. The forgetful functor $U\colon \mathbf{Games} \to \mathrm{Set}$, which is lax-monoidal, induces a monoid structure on the set UT.
3. Let $\mathcal{B}_{*,\mathbb{A}}$ be the maximum quotient monoid of UT that the canonical map $\mathrm{h}_{\mathbb{A},T}\colon UT \to A$ factors through.

The multiplication of $\mathcal{B}_{*,\mathbb{A}}$ is denoted by $\circledast$. For any game $\mathbb{X} = (X, \to)$, there is a canonical function

$$\mathcal{H}_{*,\mathbb{A},\mathbb{X}}\colon X = U\mathbb{X} \to UT \to \mathcal{B}_{*,\mathbb{A}}$$

Generalized generalized Bouton's theorem

So far, from

- a monoidal structure $*$ on **Games** such that U is lax monoidal, and
- a $\mathcal{P}_{\mathrm{fin}}$-algebra $\mathbb{A}$

we have obtained

- the Bouton monoid $(\mathcal{B}_{*,\mathbb{A}}, \circledast)$, and
- a function $\mathcal{H}_{*,\mathbb{A},\mathbb{X}} \colon X \to \mathcal{B}_{*,\mathbb{A}}$ for each game $\mathbb{X}$.

Theorem (Generalized generalized Bouton's theorem)

1. *For any game $\mathbb{X} = (X, \to)$, the function $\mathrm{h}_{\mathbb{A},\mathbb{X}}$ factors through $\mathcal{H}_{*,\mathbb{A},\mathbb{X}}$.*
2. *For two games $\mathbb{X}$ and $\mathbb{Y}$, we have*

$$\mathcal{H}_{*,\mathbb{A},\mathbb{X}*\mathbb{Y}}(x, y) = \mathcal{H}_{*,\mathbb{A},\mathbb{X}}(x) \circledast \mathcal{H}_{*,\mathbb{A},\mathbb{Y}}(y)$$

Where does the Nim-sum come from?

Proposition

Consider the case where $*$ is the Conway addition $+$, and $\mathbb{A}$ is the W/L algebra.

- *In this case, our generalized generalized Bouton's theorem coincides with the original Bouton's theorem.*
- *That is, we have $\mathcal{H}_{+,W/L,\mathbb{X}}(x) = \mathcal{G}_{\mathbb{X}}(x)$, and*
- *the Bouton monoid $\mathcal{B}_{+,W/L}$ is isomorphic to the abelian group of nim-sum $(\mathbb{N}, \oplus)$!!*

Nim-sum comes from the Conway-addition and Winning/Losing algebra!

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Other instances of the gen. gen. Bouton's theorem

By considering other monoidal structures, we want to provide new theorems to combinatorial game theory (and write a paper)!

- I have already constructed another monoidal structure,

Other instances of the gen. gen. Bouton's theorem

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Open question

Classify all monoidal structure on **Games** such that $U\colon \mathbf{Games} \to \mathrm{Set}$ is (lax) monoidal.

Categorical structure of games

Proposition

The category of games **Games**

- *is locally finitely presentable (and in particular, complete and cocomplete).*
- *has a subobject classifier.*
- *admits (Epi, Mono) factorization system.*
- *is NOT cartesian closed.*

Open question (vague)

Is there a "nice" Grothendieck topos that contains **Games** as a coreflective subcategory?

Unifying other game theories

Our approach does not depend on Set and $\mathcal{P}_{\mathrm{fin}}$, and may be applicable for a broader class of categories and endofunctors.

Open question

Generalize our framework to a (locally presentable) category and an (accessible and taut) endofunctor.

This is motivated not only by category theory, but also by game theory!

- Transfinite games ($\mathcal{P}_{<\kappa} \colon \mathrm{Set} \to \mathrm{Set}$)
- Partizan games ($\mathcal{P}_{\mathrm{fin}} \times \mathcal{P}_{\mathrm{fin}} \colon \mathrm{Set} \to \mathrm{Set}$)
- Multi-player games (?)
- Probabilistic games (?)

Differential structure on Games

Considering the origin of nim-sum, I am not fully satisfied with this result. In particular, we did mot demystify the following non-trivial equation, which was crucial to prove the original Bouton's theorem:

Proposition

$$\mathrm{mex}(S) \oplus \mathrm{mex}(T) = \mathrm{mex}((S \oplus \mathrm{mex}(T)) \cup (\mathrm{mex}(S) \oplus T)).$$

To me, this looks like the Rota-Baxter equation:

$$(\textstyle\int f) \cdot (\int g) = \int((f \cdot \int g) + (\int f \cdot g)).$$

Open question

Is there a differential (2-rig) structure on **Games**?

Open questions

Open question

- Classify all monoidal structure on **Games** such that $U\colon \mathbf{Games} \to \mathrm{Set}$ is (lax) monoidal.
- Is there a "nice" Grothendieck topos that contains **Games** as a coreflective subcategory?
- Generalize our framework to a (locally presentable) category and an (accessible and taut) endofunctor.
- Is there a differential (2-rig) structure on **Games**?

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