A Rota-Baxter equation for winning games

Games as recursive coalgebras and integrate invariants

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Differentiation in category theory and program semantics

Partially based on a joint work with Ryo Suzuki.

Slides

The (too simplified) dichotomy in this talk

Differentiation vs **Integration**

Play forward	**Analyze backward**
Coalgebra	**Algebra**
Leibniz rule	**Rota-Baxter equation**

$$\partial(xy) = (\partial x)y + x(\partial y)$$

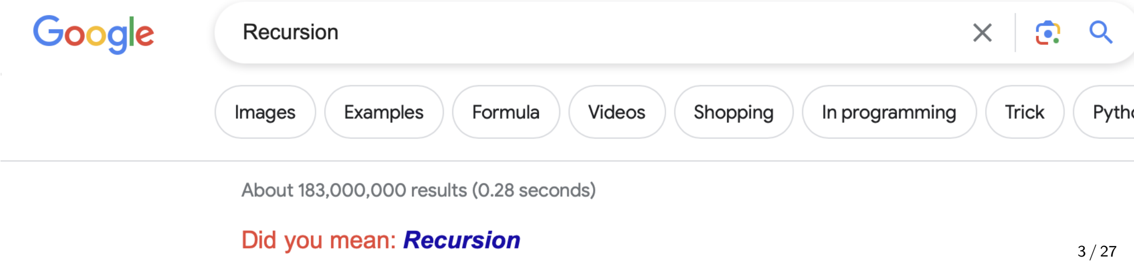
$$\left(\int f\right)\left(\int g\right) = \int \left(\left(\int f\right) g + f\left(\int g\right)\right)$$

Motivation (1/2): Game-theoretic context

We want to win games! (c.f. cyclic nim)

$\longrightarrow$ study interactions between **algebras** and **recursions**

cf. Joyal 1977; Honsell and Lenisa 2009



Motivation (2/2): Today: Providing a phenomenon

I am a beginner in this field.
This talk aims to provide a game-theoretic **phenomenon**:

Summary of this talk

The winning strategy of Nim comes from a "differential structure" on a category of "games"(:= recursive coalgebras) and an "integral structure" on $\mathcal{P}_{\text{fin}}(\mathbb{N})$.

I would appreciate any idea to make it categorical (and related works)!

- Differential category theory and its variants
- Differential 2-rig of species
- Algebraic or categorical aspects of Rota-Baxter equation
- Game semantics of (linear or some other) logic.

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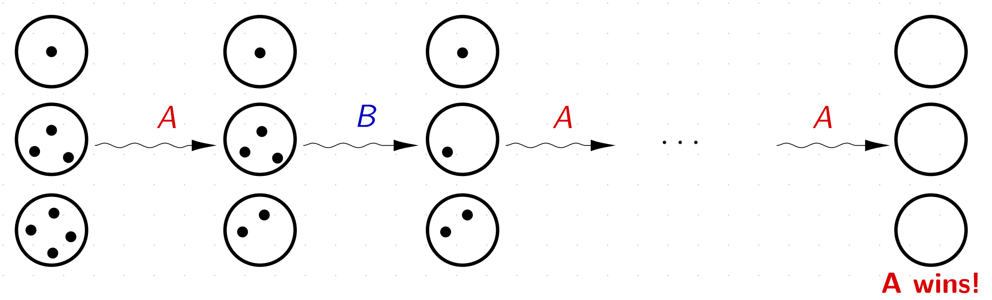
Nim (1/2): Rule of n-heap nim

- n heaps of stones are given.
- Two players take turns choosing one heap and removing at least one stone from that heap.
- The player who is unable to take a stone loses.

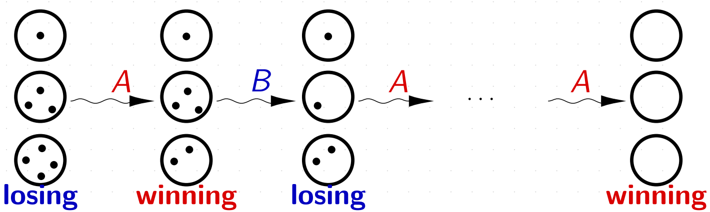


Nim(2/2): Bouton's winning strategy

Definition (Nim-sum)

The ***Nim-sum*** $\oplus$ is a binary operation on $\mathbb{N}$ defined by the"bit-wise xor."

$$
\begin{array}{rccc}
3 = & 0 & 1 & 1 \\
\oplus & & & \\
5 = & 1 & 0 & 1 \\
\hline
6 = & 1 & 1 & 0
\end{array}
$$



Theorem ([Bouton, 1901])

A state of n-heap nim $(a_1, \ldots, a_n)$ is a winning state (= "P-state") if and only if $a_1 \oplus \ldots \oplus a_n = 0$.

... where is differentiation?

Definitions (1/3): (Today's) Games

Definition ((Impartial) Game)

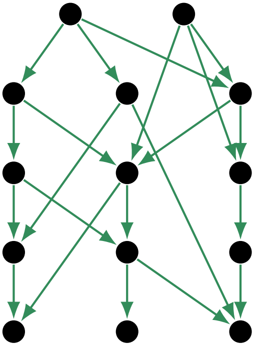
A ***game*** $\mathbb{X} = (X, \to)$ is a pair of a (possibly infinite) set X and a binary relation $\to \subset X \times X$ that satisfies the following two finiteness conditions

1. (finite options) $\#\{x' \in X \mid x \to x'\}$ is finite, for any $x \in X$.
2. (finite time) There is no infinite path. $x_0 \to x_1 \to x_2 \to \dots$

Example (Nim)

The n-heap nim $\mathrm{Nim}_n = (\mathbb{N}^n, \to)$ is defined by

$$(a_i)_{1 \le i \le n} \to (b_i)_{1 \le i \le n} \iff \exists i (a_i > b_i \wedge a_j = b_j (j \ne i))$$

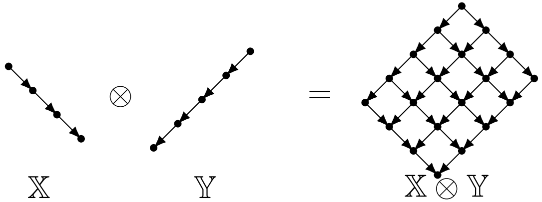


Definitions (2/3):Box product of games (+ Leibniz rule)

Definition (Box product a.k.a. Conway "addition")

The ***box product*** of two games, $\mathbb{X} = (X, \to_X)$ and $\mathbb{Y} = (Y, \to_Y)$, is the game $\mathbb{X} \otimes \mathbb{Y} = (X \times Y, \to_\otimes)$, where

- the underlying set is the cartesian product $X \times Y$, and
- the relation $\to_\otimes$ is defined by
  $(x, y) \to_\otimes (x', y') \iff (x \to_X x' \wedge y = y') \vee (x = x' \wedge y \to_Y y')$



$\partial(xy) = \partial(x)y + x\partial(y)$ ...?

Example (Nim)

$$\mathrm{Nim}_n \cong \underbrace{\mathrm{Nim}_1 \otimes \cdots \otimes \mathrm{Nim}_1}_{n}$$

Definitions (3/3): Mex, Grundy number and Rota-Baxter eq.

Definition (mex)

The ***mex*** of a finite subset $S \subset \mathbb{N}$ is $\min(\mathbb{N} \setminus S)$.

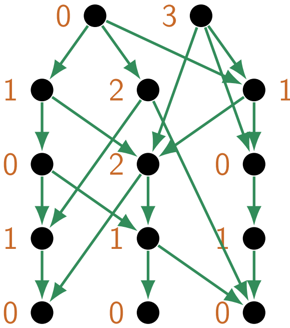
Definition (Grundy number)

For a game $\mathbb{X} = (X, \rightarrow)$, the ***Grundy number*** $\mathcal{G}_{\mathbb{X}} \colon X \to \mathbb{N}$ is recursively defined by

$$\mathcal{G}_{\mathbb{X}}(x) := \operatorname{mex}\{\mathcal{G}_{\mathbb{X}}(x') \mid x \rightarrow x'\}$$

Proposition (Grundy number is enough to win!)

For a game $\mathbb{X} = (X, \rightarrow)$, a state x is a winning state if and only if $\mathcal{G}_{\mathbb{X}}(x) = 0$.



Analyze backwards!

Rota-Baxter equation in Bouton's theorem!

Theorem (Generalized Bouton's theorem [see CGT, Siegel])

For two games $\mathbb{X} = (X, \to_X)$ and $\mathbb{Y} = (Y, \to_Y)$, we have

$$\mathcal{G}_{\mathbb{X}\otimes\mathbb{Y}}(x, y) = \mathcal{G}_{\mathbb{X}}(x) \oplus \mathcal{G}_{\mathbb{Y}}(y).$$

Sketch of the proof.

The only non-trivial part is that, for any $S, T \in \mathcal{P}_{\text{fin}}(\mathbb{N})$,
$\mathrm{mex}(S) \oplus \mathrm{mex}(T) = \mathrm{mex}((\mathrm{mex}(S) \oplus T) \cup (S \oplus \mathrm{mex}(T)))$ $\square$

This is a **Rota-Baxter equation**!

$$\left(\int f\right)\left(\int g\right) = \int \left(\left(\int f\right) g + f \left(\int g\right)\right).$$

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Coalgebras (1/2): Coalgebras and Algebras of an endofunctor

Definition (T-Algebras and T-Coalgebras)

For a category $\mathcal{C}$ and an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$,

- A T-***coalgebra*** is a pair (X, θ) of an object X of $\mathcal{C}$ and a morphism $\theta\colon X \to TX$.
- A T-***algebra*** is a pair (A, α) of an object A of $\mathcal{C}$ and a morphism $\alpha\colon TA \to A$.

Today, we consider

- $\mathcal{C} = \mathbf{Set}$ and
- $T = \mathcal{P}_{\mathrm{fin}}\colon \mathbf{Set} \to \mathbf{Set}$: The covariant finite powerset functor.

Coalgebras (2/2): Recursive coalgebra

Definition (Coalgebra-algebra morphism)

A ***coalgebra-algebra morphism*** from a T-coalgebra (X, θ) to a T-algebra (A, α) is a morphism $f : X \to A$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta"] & A \\ TX \arrow[r, "Tf"] & TA \arrow[u, "\alpha"'] \end{tikzcd}$$

Definition (Recursive coalgebra)

A T-coalgebra (X, θ) is ***recursive*** if for any T-algebra (A, α), there uniquely exists a coalgebra-algebra morphism $(X, \theta) \to (A, \alpha)$.

Games = Recursive $\mathcal{P}_{\mathrm{fin}}$-coalgebras

Definition (Games as Recursive coalgebras)

(Today,) we definie a category of games **Game** to be the category of recursive $\mathcal{P}_{\mathrm{fin}}$-coalgebras.

Obs 1. Defining $\theta(x) := \{x' \mid x \to x'\}$, recursive $\mathcal{P}_{\mathrm{fin}}$-coalgebras are exactly the "games" defined in Section 1.

Obs 2.
$$\begin{tikzcd} X \arrow[r, "\mathcal{G}_{\mathbb{X}}"] \arrow[d, "\theta"'] & \mathbb{N} \\ \mathcal{P}_{\mathrm{fin}}(X) \arrow[r, "\mathcal{P}_{\mathrm{fin}}(\mathcal{G}_{\mathbb{X}})"'] & \mathcal{P}_{\mathrm{fin}}(\mathbb{N}) \arrow[u, "\mathrm{mex}"'] \end{tikzcd} \iff \mathcal{G}_{\mathbb{X}}(x) := \mathrm{mex}\{\mathcal{G}_{\mathbb{X}}(x') \mid x' \in \theta(x)\}$$

Obs 3. Many other "game values" are induced by $\mathcal{P}_{\mathrm{fin}}$-algebras. (cf. Bai et al. 2024)

Digression: Categorical structure of games

The category of games **Game** has good categorical properties, including:

Proposition (Game is a l.f.p. s.m.c.c. w/ a subobject classifier)

- *The category of games* **Game** *is* ***locally finitely presentable****.*
  - *In particular, it is complete and cocomplete.*
- *The box product* $\otimes$ *is a symmetric monoidal closed structure on* **Game***.*
- **Game** *has a subobject classifier.*

My context: [Hora 2025]

I have given a way to calculate "game value" induced by any $\mathcal{P}_{\mathrm{fin}}$-algebra and any monoidal structure on **Game** (which makes the forgetful functor lax monoidal).
This talk is about "question 5.1" on the preprint.

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Diff. of games (1/2): Differential operator on $\mathrm{Fam}(\mathbf{Game}_*)$

Definition (Pointed games)

A ***pointed game*** is a pair $(\mathbb{X}, x)$ of a game $\mathbb{X} = (X, \to)$ and a state $x \in X$.

We write

- $\mathbf{Game}_*$ for the category of pointed games, and
- $\mathrm{Fam}(\mathbf{Game}_*)$ for the category of finite family of pointed games[1].

Definition (Differential operator)

We define ***differential operator*** $\partial \colon \mathrm{Fam}(\mathbf{Game}_*) \to \mathrm{Fam}(\mathbf{Game}_*)$ by (linearly extending)

$$\partial(\mathbb{X}, x) := \{(\mathbb{X}, x')\}_{x \to x'}.$$

[1] That is the free finite-coproduct cocompletion of $\mathbf{Game}_*$.

Diff. of games (2/2): Leibniz rule for Box product

$\mathrm{Fam}(\mathbf{Game}_*)$ has

addition $+$ = formal sum of families

multiplication $\otimes$ = (indexwise) box product.

$$\begin{tikzcd}
 & \mathbb{X} \otimes \mathbb{Y} \arrow[dl, "\text{move in } \mathbb{X}"'] \arrow[dr, "\text{move in } \mathbb{Y}"] & \\
(\partial \mathbb{X}) \otimes \mathbb{Y} & + & \mathbb{X} \otimes (\partial \mathbb{Y})
\end{tikzcd}$$

Proposition ($\mathrm{Fam}(\mathbf{Game}_*)$ forms a differential 2-rig!)

The differential operator $\partial : \mathrm{Fam}(\mathbf{Game}_*) \to \mathrm{Fam}(\mathbf{Game}_*)$ satisfying the categorified Leibniz rule:

$$\partial(\mathcal{X} \otimes \mathcal{Y}) \cong (\partial \mathcal{X}) \otimes \mathcal{Y} + \mathcal{X} \otimes (\partial \mathcal{Y}).$$

(c.f. Joyal 1981, Loregian and Trimble 2021)

Integral rigs (1/2): Definition and "Generating function"

Definition (Integral rig)

An ***integral rig*** is a rig equipped with a unary operator $\int$ with (or Rota–Baxter rig of weight 0).

$$1 = \int 0, \qquad \left(\int f\right)\left(\int g\right) = \int\left(\left(\int f\right) g + f\left(\int g\right)\right).$$

Definition ("Generating function")

Let A be an integral rig. For a pointed game (X, x), we recursively define $F_{(X,x)} \in A$ by

$$F_{(X,x)} := \int\left(\sum_{x \to_\theta x'} F_{(X,x')}\right) \in A$$

Integral rigs (2/2): F respects the rig structure.

The following "theorems" are easily proven.

Theorem (F is a rig homomorphism)

This assignment preserves the rig operations:

$$F_{\mathcal{X}\sqcup\mathcal{Y}} = F_{\mathcal{X}} + F_{\mathcal{Y}}, \qquad F_{\mathcal{X}\otimes\mathcal{Y}} = F_{\mathcal{X}} \times F_{\mathcal{Y}}.$$

Theorem

In addition, if an integral rig A is equipped with an unary operator $\partial \colon A \to A$ satisfying $\partial \int x = x$, we also have

$$\partial F_{\mathcal{X}} = F_{\partial\mathcal{X}}$$

Examples(1/3): C^∞-functions $\mathbb{R} \to \mathbb{R}$

Example ($C^\infty(\mathbb{R})$)

0	const. at 0
1	const. at 1
$+$	$+$
$\times$	$\times$
∂	$\partial f = \frac{df}{dx}$
$\int$	$\left(\int f\right)(x) := \int_0^x f(t)dt$

The "generating function" is the actual **generating function of the n-turn plays**.

$$F_{(\mathbb{X},x)}(z) = \sum_{n=0}^{\infty} \#\{x \to x_1 \to \cdots \to x_n\} \frac{z^n}{n!}$$

(This reminds me of Joyal's species...)

Naive question

Is there a nice (possibly operadic) way to **compose games**? (told in a personal comversation with Jeremie.)

Examples(2/3): Nim-sum and mex form an integral rig

The Rota-Baxter equation

$$\mathrm{mex}(S) \oplus \mathrm{mex}(T) = \mathrm{mex}\big((\mathrm{mex}(S) \oplus T) \cup (S \oplus \mathrm{mex}(T))\big)$$

is a part of the following structure.

Example ($\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$)

0	$\emptyset$
1	$\{0\}$
$+$	$\cup$
$\times$	$S \times T := \{s \oplus t \mid s \in S,\ t \in T\}$
∂	does not exist
$\int$	$\int(S) := \{\mathrm{mex}(S)\}$

Theorem (mex as integration)

This makes $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$ into an integral rig.

(By post-composing $\mathcal{P}_{\mathrm{fin}}(\mathbb{N}) \to \mathbb{B} := \{\bot, \top\}$) we have

Corollary

Bouton's winning strategy of Nim!

Examples(3/3): Hereditarily finite sets

Let V_ω be the set of all ***hereditarily finite sets***.

Example ($\mathcal{P}_{\mathrm{fin}}(V_\omega)$)

0	$\emptyset$
1	$\{\emptyset\}$
$+$	$\cup$
$\times$	complicated
∂	$\partial A = \bigcup_{B \in A} B$
$\int$	$\int A = \{A\}$

Remark(Why $\mathcal{P}_{\mathrm{fin}}(V_\omega)$? 1)

$$\begin{aligned} V_\omega &= \text{the initial } \mathcal{P}_{\mathrm{fin}}\text{-algebra} \\ &= \text{the terminal recursive } \mathcal{P}_{\mathrm{fin}}\text{-coalgebra} \end{aligned}$$

Remark(Why $\mathcal{P}_{\mathrm{fin}}(V_\omega)$? 2)

For any monoid M, there is a bij. corresp. between

- differential operator ∂ on the rig $\mathcal{P}_{\mathrm{fin}}(M)$, and
- **Game**-enrichment(*) of M

This talk in One slide

1. Box product satisfies the Leibniz rule, and Nim-sum satisfies the Rota–Baxter equation.
2. Games as recursive coalgebras, and Grundy numbers[2] as the unique colagebra-algebra morphism.
3. **The classical winning strategy of Nim follows from those calculus structures.**

Questions

Connections with differential categories or species?

[2] or any other "recursively defined" value.

References

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