Local state classifier for automata theory (IRIF)

Ryuya Hora

University of Tokyo

May 27, 2025

Slides

- *Topoi of automata I* [Hora 2024b]
- *Topoi of automata II* [Hora n.d.]
- *Internal Parameterization of Hyperconnected Quotients* [Hora 2024a]

Table of Contents

Table of Contents

Original Motivation: Lawvere's open problems in topos theory

In 2009, W.Lawvere posed 7 open problems in topos theory [Lawvere 2009]:

1. **Quotient Toposes** [Kamio and Hora 2024]
2. Subqotients and Idempotents
3. Boundaries of classes of models
4. The jump operator on levels within a topos [Hora, Kamio, and Maehara 2025]
5. Coverings that admit averaging and microlinearity
6. How strong is the adjointness of fractional exponents?
7. The algebra of time

I defined **local state classifiers** as a partial[1] solution to the first [Hora 2024a]. However, I will motivate LSC from another point of view today.

1

Today's motivation

The aim of this talk is to explain the following sentence, which is the calculational—rather than conceptual—starting point of a topos-theoretic approach to automata theory.

Summary of this talk

Word congruences form the **local state classifier** Ξ of Σ-**Set**.

LSCs	2	Define LSC Ξ as a colimit of all monomorphisms.
	3	Get used to this abstract concept through many examples.
Atmt	4	Observe that LSC Ξ of Σ-**Set** consists of word congruences.
	5	Provide one example of calculations with LSC.

Table of Contents

Definition of LSC

Definition (local state classifier [Hora 2024a])

A **local state classifier** Ξ of a category $\mathcal{C}$ is a colimit of all monomorphisms:

$$\Xi = \operatorname{colim}(\mathcal{C}_{\text{mono}} \rightarrowtail \mathcal{C}).$$

The associated cocone is referred to as $\{\xi_X : X \to \Xi\}_{X \in \mathrm{ob}\mathcal{C}}$.

$$\begin{tikzcd} U \arrow[rr, tail, "\iota"] \arrow[dr, "\xi_U"'] & & X \arrow[dl, "\xi_X"] \\ & \Xi & \end{tikzcd}$$

Toy Example (1/3): sets

Example (Set)

The local state classifier of **Set** is the terminal object $\Xi = \{*\}$.

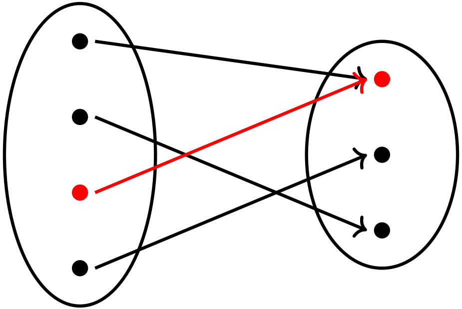
Consider an arbitrary cocone $\{\alpha_X : X \to L\}$ of $\mathbf{Set}_{\mathrm{inj}} \rightarrowtail \mathbf{Set}$. Then, for any set X and its element $x \in X$, we have

$$
\begin{tikzcd}
\{*\} \arrow[rr, tail, "\mathsf{pick}_x"] \arrow[dr, "\alpha_{\{*\}}"'] & & X \arrow[dl, "\alpha_X"] \\
& L &
\end{tikzcd}
$$

and therefore $\alpha_X(x) = \alpha_{\{*\}}(*)$.

Toy Example (2/3): pointed sets

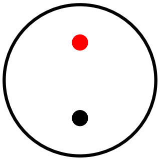
What is the local state classifier of the category of pointed sets $\mathbf{Set}_*$?



Toy Example (2/3): pointed sets

What is the local state classifier of the category of pointed sets $\mathbf{Set}_*$?
It is

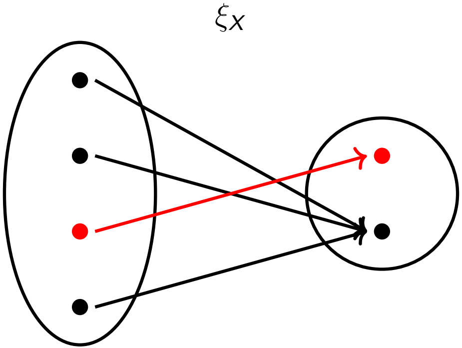
Ξ



equipped with ...

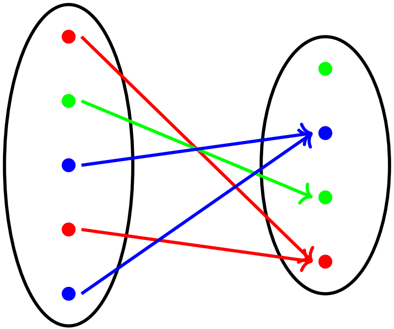
Toy Example (2/3): pointed sets

cocone maps



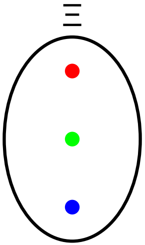
Toy Example (3/3): colored sets

What is the local state classifier of the category of 3-colored sets **Set**$/\{R,G,B\}$?



Toy Example (3/3): colored sets

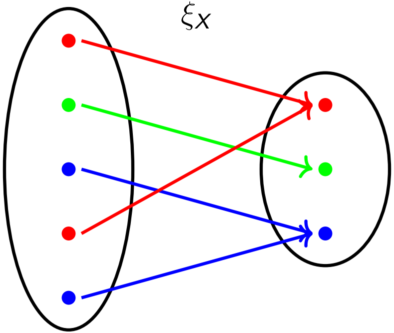
What is the local state classifier of the category of 3-colored sets **Set**/{R,G,B}?
It is



equipped with ...

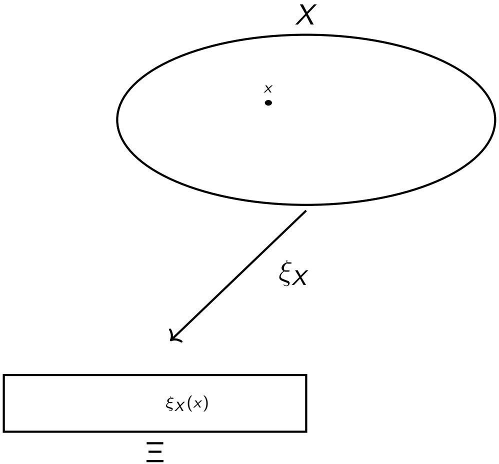
Toy Example (3/3): colored sets

morphisms that classify elements by their colors.



Intuition: why local?

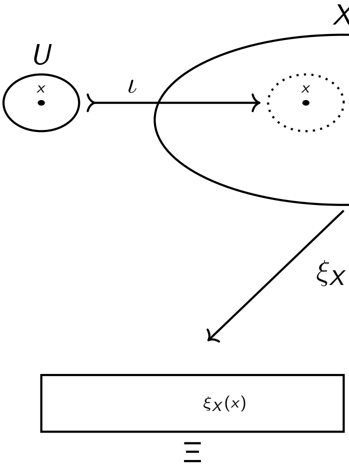
Let $\{\xi_X \colon X \to \Xi\}_{X \in \mathrm{ob}\mathcal{C}}$ be a local state classifier.



The value of $\xi_X(x)$ can be calculated in arbitraly small neighborhood of x.

Intuition: why local?

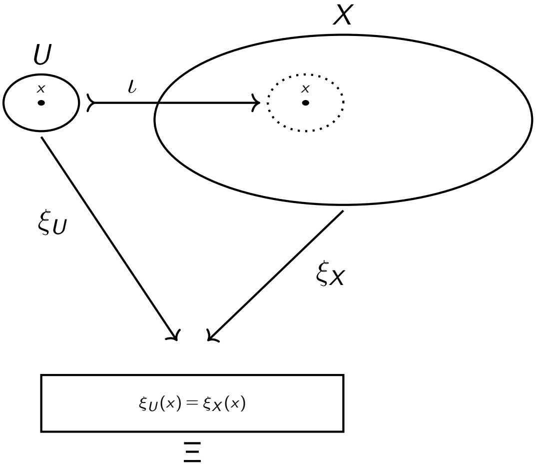
Let $\{\xi_X : X \to \Xi\}_{X \in \mathrm{ob}\mathcal{C}}$ be a local state classifier.



The value of $\xi_X(x)$ can be calculated in arbitraly small neighborhood of x.

Intuition: why local?

Let $\{\xi_X : X \to \Xi\}_{X \in \mathrm{ob}\mathcal{C}}$ be a local state classifier.

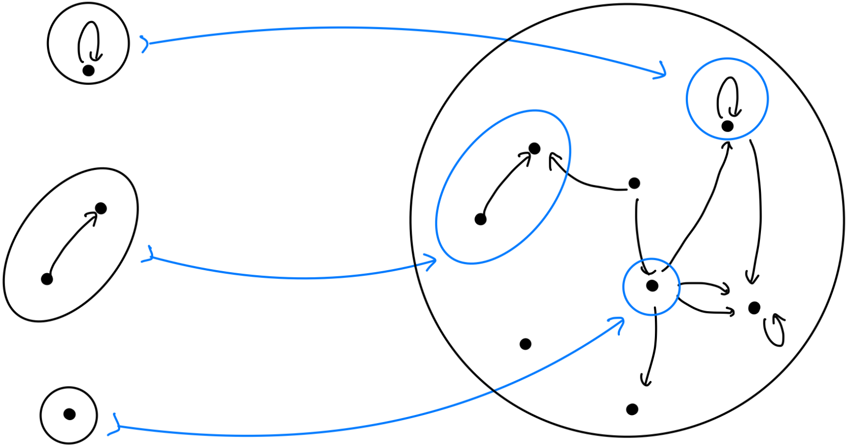


The value of $\xi_X(x)$ can be calculated in arbitraly small neighborhood of x.

Table of Contents

More Examples (1/4): directed graphs

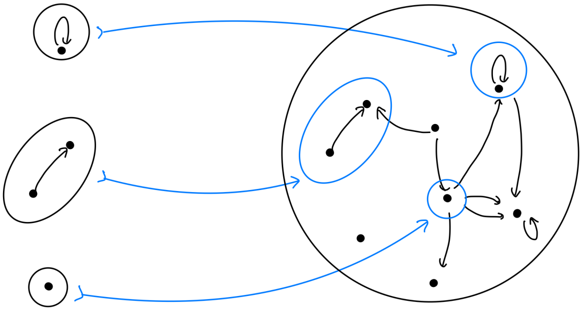
What is the local state classifier of the category of directed graphs $\mathbf{PSh}(\rightrightarrows)$?



More Examples (1/4): directed graphs

The local state classifier of the topos of directed graphs $\mathbf{PSh}(\rightrightarrows)$ is given by the "2-bouquet."

1. Ξ is [being a loop]  [not being a loop].
2. ξ_X detects whether a given edge is a loop or not.



More Examples (2/4): group actions for a group G

What is the local state classifier of $\mathbf{PSh}(G)$?
$=$ What is the "local data" for each element of a G-set $X \curvearrowleft G$?

More Examples (2/4): group actions for a group G

What is the local state classifier of $\mathbf{PSh}(G)$?
= What is the "local data" for each element of a G-set $X \curvearrowleft G$?
For the category of right G-actions $\mathbf{PSh}(G)$ for a group G,

1. Ξ is the set of all subgroups $\mathrm{Sub}_{\mathrm{Grp}}(G)$ (equipped with the conjugate action).
2. ξ_X sends each element $x \in X$ to its stabilizer subgroup.

More Examples (3/4): presheaves

Let $\mathcal{C}$ be a small category. The local state classifier of the presheaf topos $\mathbf{PSh}(\mathcal{C})$ is the presheaf of quotient objects of representable presheaves:

$$\Xi\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}\colon c \mapsto \{y(c) \twoheadrightarrow C\}$$

More Examples (3/4): presheaves

Let $\mathcal{C}$ be a small category. The local state classifier of the presheaf topos $\mathbf{PSh}(\mathcal{C})$ is the presheaf of quotient objects of representable presheaves:

$$\Xi \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set} \colon c \mapsto \{y(c) \twoheadrightarrow C\}$$

Remark (Dual similarity with the Subobject classifier)

It is a kind of dual to the subobject classifier, which is the presheaf of subobjects of representable presheaves:

$$\Omega \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set} \colon c \mapsto \{y(c) \leftarrowtail C\}$$

More Examples (4/4): Other categories

The following categories have LSCs.

- The category of simplicial sets **sSet**
- The topos of Joyal's species $\mathbf{FinSet}^{\mathbf{FinSet}_0}$ (cf. [Fiore, Galal, and Paquet 2024])
- The category of coalgebraic automata $\mathbf{Coalg}_{2\times^{\Sigma}} = \mathbf{Atmt}_{\Sigma}$.
- The Schanuel topos $\mathbf{Cont}(\mathrm{Aut}_{\mathbf{Set}}(\mathbb{N})) \cong \mathbf{Sh}(\mathbf{FinSet}_{\mathrm{mono}}, J_{\mathrm{at.}})$
- **Ab**, **SemiLattice**, and any other categories of algebras.
- Any Grothendieck topoi including $\mathbf{Sh}(X)$ for a topological space X.

Table of Contents

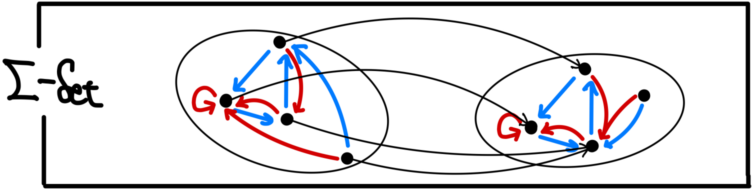
Σ-sets

We fix a finite set Σ. A Σ-**set** is a pair (Q, δ) of a set Q and a function $\delta\colon Q \times \Sigma \to Q$.

Definition

Σ-**Set** is defined by $\Sigma\text{-}\mathbf{Set} := \mathbf{PSh}(\Sigma^*)$.

$\Sigma = \{a, b\}$



The Σ-set of right congruences

A **right congruence** is an equivalence relation $\sim$ on Σ^* such that

$$\forall u, v, w \in \Sigma^*,\ u \sim v \implies uw \sim vw.$$

Definition (The Σ-set of right congruences Ξ)

Element Right congruence ($=$ quotient object of the representable Σ^*.)

Σ-action $u \ (\sim * w)\ v \iff wu \sim wv.$

Right congruences form LSC

For any Σ-set (Q, δ), we define a morphism

$$\xi_{(Q,\delta)} \colon (Q, \delta) \to \Xi$$

by sending $q \in Q$ to the right congruence $\xi_{(Q,\delta)}(q) \in \Xi$:

$$u \; \xi_{(Q,\delta)}(q) \; v \iff q * u = q * v .$$

Proposition (LSC of Σ-Set, Universality of right congruences)

*The Σ-set of right congruences Ξ, together with the cocone $\{\xi_{(Q,\delta)} \colon (Q, \delta) \to \Xi\}_{(Q,\delta)\in \mathrm{ob}(\Sigma\text{-}\mathbf{Set})}$ is the local state classifier of Σ-**Set**.*

Toy example of LSC calculation

Example (The Σ-set of languages $\mathbf{Lan}$)

Element	Language
Σ-action	$L * w := \{v \in \Sigma^* \mid wv \in L\}$ (Brzozowski derivative)
Universality	$\Sigma\text{-}\mathbf{Set}((Q,\delta),\mathbf{Lan}) \cong \mathcal{P}(Q)$

What's the cocone component $\xi_{\mathbf{Lan}} \colon \mathbf{Lan} \to \Xi$?

Toy example of LSC calculation

Example (The Σ-set of languages Lan)

Element	Language
Σ-action	$L * w := \{v \in \Sigma^* \mid wv \in L\}$ (Brzozowski derivative)
Universality	$\Sigma\text{-}\mathbf{Set}((Q,\delta), \mathbf{Lan}) \cong \mathcal{P}(Q)$

What's the cocone component $\xi_{\mathbf{Lan}} \colon \mathbf{Lan} \to \Xi$?

Proposition (A categorical derivation of Nerode congruence)

For any language L, $\xi_{\mathbf{Lan}}(L)$ coincides with the ***Nerode congruence*** *of L.*

$$u \; \xi_{\mathbf{Lan}}(L) \; v \iff L * u = L * v \iff (\forall w \in \Sigma^* \; uw \in L \iff vw \in L)$$

Table of Contents

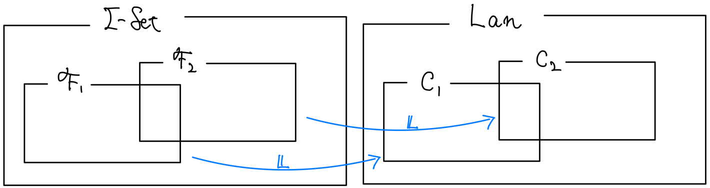
Idea: Σ-set classes → language classes

From a full subcategory $\mathcal{F} \subset \Sigma\text{-}\mathbf{Set}$, we can consider the **induced language class** $\mathbb{L}_{\mathcal{F}} \subset \mathbf{Lan}$.

$$\mathbb{L}_{\mathcal{F}} := \{L \in \mathbf{Lan} \mid \text{some } \Sigma\text{-set in } \mathcal{F} \text{ recognizes } L\}.$$

Example (regular languages)

$\Sigma\text{-}\mathbf{FinSet}$ is the category of finite Σ-sets. $\mathbb{L}_{\Sigma\text{-}\mathbf{FinSet}}$ is the set of regular languages.



Q: What kind of full subcategories should we consider?

It suffices to consider **nice full subcategories.** but which?
In category theory, we tend to consider reflective subcategories:

- Subtopoi? (= lex reflective subcat)
- Birkhoff's HSP varieties? (= subquotient-closed reflective sub)
- Orthogonality classes?

[2]This is the reason why it's related to the open problem of **quotient topoi**.

Q: What kind of full subcategories should we consider?

It suffices to consider **nice full subcategories.** but which?
In category theory, we tend to consider reflective subcategories:

- Subtopoi? ($=$ lex reflective subcat)
- Birkhoff's HSP varieties? ($=$ subquotient-closed reflective sub)
- Orthogonality classes?

We need to go the "opposite" direction[2], coreflective ones!

[2] This is the reason why it's related to the open problem of **quotient topoi.**

A: Covariety of Σ-**Set**

Definition (Covariety)

A **covariety**[3] of Σ-**Set** is a full subcategory of Σ-**Set** that is closed under

- subobjects,
- quotient objects, and
- coproducts.

Every covariety is **coreflective.**

Lemma (Covarieties are enough.)

*For any full subcategory $\mathcal{F} \subset \Sigma$-**Set**, there exists the minimum covariety $\overline{\mathcal{F}} \subset \Sigma$-**Set** that contains $\mathcal{F}$. Furthermore, we have $\mathbb{L}_{\mathcal{F}} = \mathbb{L}_{\overline{\mathcal{F}}}$.*

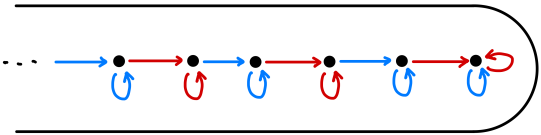
[3] This terminology is due to Peter Johnstone. This is defined for any Grothendieck topos.

Example: What is $\overline{\Sigma\text{-}\mathbf{FinSet}}$? Orbit-finiteness

Definition (Σ-$\mathbf{Set}_{\mathrm{o.f.}}$)

A Σ-set $(Q, \delta \colon Q \times \Sigma \to Q)$ is said to be **orbit-finite** if, for any state $q \in Q$, its orbit $q\Sigma^* := \{qw \mid w \in \Sigma^*\}$ is a finite set.

Let Σ-$\mathbf{Set}_{\mathrm{o.f.}}$ denote the full subcategory of Σ-**Set** that consists of all orbit-finite Σ-sets.



Example

$\overline{\Sigma\text{-}\mathbf{FinSet}} = \Sigma\text{-}\mathbf{Set}_{\mathrm{o.f.}} \hookrightarrow \Sigma\text{-}\mathbf{Set}$ is a covariety such that $\mathbb{L}_{\Sigma\text{-}\mathbf{Set}_{\mathrm{o.f.}}} = \mathbf{Reg}$.

covariety classification (1/2)

But how can we calculate covarieties of Σ-**Set**? (Even the smallness of the number of them is non-trivial a priori.)

covariety classification (1/2)

But how can we calculate covarieties of Σ-**Set**? (Even the smallness of the number of them is non-trivial a priori.)

Theorem (Classification of covarieties)

For any Grothendieck topos $\mathcal{E}$, (in particular $\mathcal{E} = \Sigma\text{-}\mathbf{Set}$,) there is a bijective correspondence between

- *covarieties of $\mathcal{E}$, and*
- *upward closed[a] subobjects $U \rightarrowtail \Xi$.*

This theorem reduces the study of covarieties of Σ-**Set** to the word combinatorics of congruences Ξ.

[a] Ξ admits a unique order structure such that $\xi_X(x) \wedge \xi_Y(y) = \xi_{X \times Y}(x, y)$.

covariety classification (2/2)

For a given upward closed subobject $U \rightarrowtail \Xi$, the corresponding covariety $\mathcal{F}_U$ is given by

$$\mathrm{ob}(\mathcal{F}_U) := \left\{ X \in \mathrm{ob}(\mathcal{E}) \;\middle|\; \begin{tikzcd} & U \arrow[d, tail] \\ X \arrow[ur, dashed, "\exists"] \arrow[r, "\xi_X"'] & \Xi \end{tikzcd} \right\}.$$

Example (Σ-$\mathbf{Set}_{\mathrm{o.f.}} \xleftrightarrow{\text{corresponding}} U_{\mathrm{o.f.}}$)

The covariety Σ-$\mathbf{Set}_{\mathrm{o.f.}}$ corresponds to the subobject $U_{\mathrm{o.f.}} := \{\sim \in \Xi \mid \Sigma^*/\sim \text{ is finite}\}$.

What I cannot talk: Other examples of calculation with LSC

Ξ translate abstract definitions into concrete word conbinatorics:

What I cannot talk: Other examples of calculation with LSC

Ξ translate abstract definitions into concrete word conbinatorics:

- **Myhill-Nerode theorem** $\mathbb{L}_{\mathcal{F}} = \xi_{\mathbf{Lan}}^{-1}(U)$ for each covariety $\mathcal{F}$.
  - $\mathbb{L}_{\Sigma\text{-}\mathbf{Set}_{\mathrm{o.f.}}} = \xi_{\mathbf{Lan}}^{-1}(U_{\mathrm{o.f.}})$ (cf. $\xi_{\mathbf{Lan}} \colon \mathbf{Lan} \to \Xi$ is the nerode congruence.)

What I cannot talk: Other examples of calculation with LSC

Ξ translate abstract definitions into concrete word conbinatorics:

- **Myhill-Nerode theorem** $\mathbb{L}_{\mathcal{F}} = \xi_{\mathbf{Lan}}^{-1}(U)$ for each covariety $\mathcal{F}$.
  - $\mathbb{L}_{\Sigma\text{-}\mathbf{Set}_{\mathrm{o.f.}}} = \xi_{\mathbf{Lan}}^{-1}(U_{\mathrm{o.f.}})$ (cf. $\xi_{\mathbf{Lan}} \colon \mathbf{Lan} \to \Xi$ is the nerode congruence.)
- **Syntactic congruence** $\cong_L = \bigwedge_{w \in \Sigma^*} (\xi_{\mathbf{Lan}}(L) * w)$ in Ξ.
  - We need $\xi_{\Xi} \colon \Xi \to \Xi$ to capture the behavior of $\xi_{\mathbf{Lan}}(L)$.

What I cannot talk: Other examples of calculation with LSC

Ξ translate abstract definitions into concrete word conbinatorics:

- **Myhill-Nerode theorem** $\mathbb{L}_{\mathcal{F}} = \xi_{\mathbf{Lan}}^{-1}(U)$ for each covariety $\mathcal{F}$.
  - $\mathbb{L}_{\Sigma\text{-}\mathbf{Set}_{\mathrm{o.f.}}} = \xi_{\mathbf{Lan}}^{-1}(U_{\mathrm{o.f.}})$ (cf. $\xi_{\mathbf{Lan}} \colon \mathbf{Lan} \to \Xi$ is the nerode congruence.)
- **Syntactic congruence** $\cong_L = \bigwedge_{w \in \Sigma^*} (\xi_{\mathbf{Lan}}(L) * w)$ in Ξ.
  - We need $\xi_{\Xi} \colon \Xi \to \Xi$ to capture the behavior of $\xi_{\mathbf{Lan}}(L)$.
- **Prodiscrete syntactic monoid** $\pi_1(C)$ for each $C \subset \mathbf{Lan}$.
  - $\pi_1(\mathbf{Reg}) = \widehat{\Sigma^*}$: the profinite monoid of profinite words
  - $\pi_1(\{L\}) = \Sigma^*/{\cong_L} \impliedby \xi_{\Xi}(\xi_{\mathbf{Lan}}(L)) \in U_{\mathrm{o.f.}} \impliedby \xi_{\mathbf{Lan}}(L) \in U_{\mathrm{o.f.}}$
  - $\pi_1(\{\mathrm{well\text{-}parenthesized}\}) = \langle a, b \mid ab = 1 \rangle \sqcup \{*\}$ with a non-trivial topology.

What I cannot talk: Other examples of calculation with LSC

Ξ translate abstract definitions into concrete word conbinatorics:

- **Myhill-Nerode theorem** $\mathbb{L}_{\mathcal{F}} = \xi_{\mathbf{Lan}}^{-1}(U)$ for each covariety $\mathcal{F}$.
  - $\mathbb{L}_{\mathbf{\Sigma\text{-}Set}_{\mathrm{o.f.}}} = \xi_{\mathbf{Lan}}^{-1}(U_{\mathrm{o.f.}})$ (cf. $\xi_{\mathbf{Lan}} \colon \mathbf{Lan} \to \Xi$ is the nerode congruence.)
- **Syntactic congruence** $\cong_L = \bigwedge_{w \in \Sigma^*} (\xi_{\mathbf{Lan}}(L) * w)$ in Ξ.
  - We need $\xi_{\Xi} \colon \Xi \to \Xi$ to capture the behavior of $\xi_{\mathbf{Lan}}(L)$.
- **Prodiscrete syntactic monoid** $\pi_1(C)$ for each $C \subset \mathbf{Lan}$.
  - $\pi_1(\mathbf{Reg}) = \widehat{\Sigma^*}$: the profinite monoid of profinite words
  - $\pi_1(\{L\}) = \Sigma^*/\cong_L \impliedby \xi_{\Xi}(\xi_{\mathbf{Lan}}(L)) \in U_{\mathrm{o.f.}} \impliedby \xi_{\mathbf{Lan}}(L) \in U_{\mathrm{o.f.}}$
  - $\pi_1(\{\text{well-parenthesized}\}) = \langle a, b \mid ab = 1 \rangle \sqcup \{*\}$ with a non-trivial topology.
- **Syntactic topos** $\mathcal{T}_C$ for each $C \subset \mathbf{Lan}$.
  - Classification of hyperconnected quotient topoi ($=$ covarieties closed under finite products)[Hora 2024a]
  - semi-Galois theory $\mathcal{T}_C \simeq \mathbf{Cont}(\pi_1(C))$ (cf. [Uramoto 2025])

References

- Fiore, Marcelo, Zeinab Galal, and Hugo Paquet (2024). "Stabilized profunctors and stable species of structures". In: *Logical Methods in Computer Science* 20.
- Hora, Ryuya (2024a). "Internal parameterization of hyperconnected quotients". In: *Theory and Applications of Categories* 42.11, pp. 263–313.
- — (2024b). "Topoi of automata I: Four topoi of automata and regular languages". In: *arXiv:2411.06358*.
- — (n.d.). *Topoi of automata II: Hyperconnected geometric morphisms, syntactic monoids, and language classes (ongoing)*.
- Hora, Ryuya, Yuhi Kamio, and Yuki Maehara (2025). *Lawvere's fourth open problem: Levels in the topos of symmetric simplicial sets*. arXiv: 2503.03439 [math.CT]. URL: https://arxiv.org/abs/2503.03439.
- Kamio, Yuhi and Ryuya Hora (2024). "A solution to the first Lawvere's problem A Grothendieck topos that has a proper class many quotient topoi". In: *arXiv:2407.17105*.
- Lawvere, F. William (2009). *Open problems in topos theory*. Can be found via "https://lawverearchives.com".
- Uramoto, Takeo (2025). "Semi-galois Categories I: The classical Eilenberg variety theory". In: *Journal of Pure and Applied Algebra* 229.2, p. 107863.