

★ 7② Birkhoff $\mathcal{C}^n$.

Dold - Kan correspondence

: Chain Complex is "Simplicial Module"

7.4.11

• simplicial set σ $\leftrightarrow$ II $\mathcal{C}^n$ is.

homology $\mathcal{C}^n$ chain complex is simplicial object
in $\mathcal{C}^n$. , Dold - Kan correspondence

$$\text{Mod}_R^{\Delta^{\text{op}}} \simeq \text{Ch}_2(\text{Mod}_R) \quad \text{is true.}$$

7.4.12

• $\mathcal{C}^n$. $C: \text{Mod}_R^{\Delta^{\text{op}}} \longrightarrow \text{Ch}_2(\text{Mod}_R)$

is homology $\mathcal{C}^n$. C is "normalized" $\mathcal{C}^n$. N is $\mathcal{C}^n$,
 $C = N \oplus D$ is $\mathcal{C}^n$. $N: \text{Mod}_R^{\Delta^{\text{op}}} \longrightarrow \text{Ch}_2(\text{Mod}_R)$

$$C = N \oplus D \quad \text{is } \mathcal{C}^n. \quad N: \text{Mod}_R^{\Delta^{\text{op}}} \longrightarrow \text{Ch}_2(\text{Mod}_R)$$

is equiv $\mathcal{C}^n$.

• N is chain complex is "semi-simplicial" module
is $\mathcal{C}^n$ is $\mathcal{C}^n$ is $\mathcal{C}^n$.

§ 1. Singular homology

§ 1.1 Singular homology

§ 1.2 unnormalized chain complex C .

§ 2. Normalized Chain Complex N .

§ 2.1 as a subfactor of C .

§ 2.2 as a "non-degenerate part" of C .

§ 2.3 as a coreflector of $Ch_2 \rightarrow Mod_R^{\Delta_{mono}^{op}}$

(§ 2.4 as a Yoneda extension)

§ 3. Dold - Kan Correspondence

§ 3.1 semi-simplicial $\rightsquigarrow$ simplicial

§ 3.2 Proof of Dold - Kan Correspondence.

§ 4. $\mathcal{C} \cong \mathcal{D} \circ \mathcal{B} \circ \mathcal{A} \circ \mathcal{C}$.

- as a Nerve and Realization

- Preservation of homotopy

↑ correspondence of monoidal structures

- $N \rightarrow C$: homotopy equiv.

§ 1.2 unnormalized chain complex C .

≡ "2". Construct $C \in \mathcal{C} \subset \mathcal{C} \subset \mathcal{C}$.

Def ("unnormalized" chain complex)

A : simplicial R -mod $\in \mathcal{C}$.

chain complex $CA = (CA_n, \partial_{n+1}: A_{n+1} \rightarrow A_n)_{n=0}^{\infty} \in \mathcal{C}$

$$\begin{cases} CA_n := A_n \\ \partial_n := \sum_{i=0}^n (-1)^i d_i^M \end{cases}$$

$\in \mathcal{C}$.

(well-defined) $\partial^2 = 0 \in \mathcal{C}$.

$\mathcal{C} \subset \mathcal{C} \subset \mathcal{C}$

$x \in A_{n+1}$ ($n \geq 1$) $\in \mathcal{C}$.

$$(\partial_n \circ \partial_{n+1})(x) = \sum_{j=0}^n (-1)^j \left(\sum_{i=0}^{n+1} (-1)^i x * \partial_i^{n+1} \right) * \partial_j^n$$

$$= \sum_{0 \leq i \leq j \leq n} (-1)^{j+i} x * \partial_{j+1}^{n+1} * \partial_i^n$$

$\mathcal{C} \subset \mathcal{C}$

$$+ \sum_{0 \leq j < i \leq n+1} (-1)^{j+i} x * \partial_i^{n+1} * \partial_j^n$$

$$= 0$$

$$\mathcal{C}, \partial^2 = 0$$

これは. functor $C : \text{Mod}_R^{\Delta^{\text{op}}} \longrightarrow \text{Ch}_2(\text{Mod}_R)$

に定まっている。

Remark

これは. equiv ではない。

実際, $A : \text{simplicial } R\text{-mod} \hookrightarrow \text{Ch}_2$.

$$[1] \xrightleftharpoons[\sigma'_1]{\partial'_1} [2] \quad \text{is} \quad A_1 \xrightleftharpoons[s'_1]{d'_1} A_2 : \text{retract}$$

is induced by $\hookrightarrow \text{Ch}_2$. A_1 is A_2 's direct summand! (if $\partial'_1 \neq 0$!)

Chain Complex ではない. 例として,

$$\begin{array}{ccccccc} \rightarrow & 0 & \rightarrow & \mathbb{Z} & \rightarrow & 0 & \\ & \color{blue}{2} & & \color{blue}{1} & & \color{blue}{0} & \end{array} \quad \text{is } \text{Im } C \text{ is } \lambda, \text{ is } \lambda, \text{ is } \lambda,$$

$\hookrightarrow C$ is essentially surj in Ch_2 .

Simplicial $R\text{-mod}$ is degenerate 部分 $\neq \Delta$ 2-2.

"1" "1" "1" is direct summand $\neq \Delta$ 2-2.

is $\hookrightarrow$ degenerate 部分 $\neq \Delta$ 2-2

"Normalized" version, $NA \in \text{Ch}_2$ (2-2).

$$\left(\begin{array}{l} \text{Ex. 1. } A_n \cong \bigoplus_{[n] \xrightarrow{\nabla} [r]} NA_r \quad \Leftrightarrow \quad \text{non-deg} \\ \leadsto \text{Analogous to } X_n \cong \bigsqcup_{[n] \xrightarrow{\nabla} [r]} NX_r \end{array} \right)$$

§ 2. Normalized Chain Complex N .

§ 2.1 as a subfunctor of C .

可. 一番大分 数の中 定数 $\mathbb{Z}(2\mathbb{Z})$.

Def (Normalized chain complex)

$$N: \text{Mod}_R^{\Delta^{\text{op}}} \longrightarrow \text{Ch}_2(\text{Mod}_R)$$

$$\varepsilon. (NA)_n := \bigcap_{i=0}^{n-1} \text{Ker}(d_i^n) \quad (\subseteq A_n)$$

$$\partial_n := (-1)^n d_n^n \quad \left(\begin{array}{l} \text{= 0 である} \\ \sum_{i=0}^n (-1)^i d_i^n \text{ は } NA_n = 0 \text{ である} \end{array} \right)$$

ε は C の subfunctor である.

well-defined

C : well-defined である.

$$\left\{ \begin{array}{l} \bullet d_n^n(NA_n) \subseteq NA_{n-1} \end{array} \right.$$

$$\left\{ \begin{array}{l} \bullet f: A \rightarrow B \quad \text{ならば} \quad f_n(NA_n) \subseteq NB_n \end{array} \right.$$

check する. simplicial identity & naturality について.

($\rightarrow$ 可. N is equiv $\mathbb{Z}(2\mathbb{Z})$)

e.g.) NA_0 は $1 \notin A_0$.

e.g.) A : constant S. module #3,

$$NA_n = \begin{cases} A & (n=0) \\ 0 & (n \geq 1), \end{cases}$$

他の計算は後で.

N は 何番か?

$\sum \#2^k < .$

§ 2.2 as a "non-degenerate part" of C .

N.B. C a "non-degenerate part" と言っているのは、

$\exists \alpha \in C$ である。

Remark

本稿では A_n の non-degenerate 部分集合 Σ_n について、
 Σ_n module について。 (これは 0 は degenerate である。)

simplicial set である。 X_n の degenerate 部分 D_n
は non-degenerate 部分 N_n である。 X_n 直和分解
 $X_n = N_n \sqcup D_n$ である。

今、 Σ_n の "non-degenerate part" である
意味は、 $\text{Mod } \Sigma_n$ 直和分解、

$$A_n = N A_n \oplus D A_n \quad \text{である。}$$

"non-degenerate" である。

"degenerate" ではない。

Def (degenerate chain complex)

$$D: \text{Mod}_R^{\Delta^{\text{op}}} \longrightarrow \text{Ch}_2(\text{Mod}_R) \quad \mathbb{E},$$

$$DA_n := \sum_{k=0}^{n-1} \text{Im}(S_k^{n-1}) \quad (\subseteq A_n)$$

これは C の subfactor である.

(well-defined)

まず, DA_n は ∂ の作用を $\partial^2 = 0$ であることと一致する.

よく見ると.

$$x \in A_{n-1}, \quad 0 \leq k \leq n-1 \quad \text{と} \quad \mathbb{Z}.$$

$$\partial(\underbrace{x * \sigma_k^{n-1}}_{\text{これは } DA_n \text{ である}}) = \sum_{i=0}^n (-1)^i x * \sigma_k^{n-1} * \partial_i^n$$

これは DA_n である
と見られる

$$= \sum_{i=0}^{k-1} (-1)^i x * \partial_i^{n-1} * \sigma_{k-1}^{n-2}$$

$$+ \underbrace{(-1)^k x + (-1)^{k+1} x}_{\text{これは!}} \quad //^0$$

$$+ \sum_{i=k+2}^n (-1)^i x * \partial_{i-1}^{n-1} * \sigma_k^{n-2}$$

$$\in DA_{n-1} \quad \#.$$

linearity は 容易である.

$= k^2$, C a subfactor $N \subseteq D$ $2^{\mathbb{Z}} \subseteq \mathbb{Z}$, $\overline{k} \in \mathbb{Z}$ $2^{\mathbb{Z}} \subseteq \mathbb{Z}$.
 $= k$. $N A_n \cong A_n / D A_n$: "degenerate mod degenerates"

Prop (N is the "non-degenerate part" of C)

A : simplicial R -mod $(2^{\mathbb{Z}} \subseteq \mathbb{Z})$.

$$A_n = N A_n \oplus D A_n \quad (n \geq 0)$$

(proof)

3.1.3

inner
 to say M_n (ordered) ~~direct sum~~ $\mathbb{Z}^n$. M is n idempotent is
 $-25-12$ ~~direct sum~~ $(2^{\mathbb{Z}} \subseteq \mathbb{Z})$.

$$\begin{array}{c} \mu = A \oplus B \\ \text{kerf} \oplus \text{Imf} \end{array} \quad \begin{array}{c} \text{wavy lines} \\ \text{crossing} \end{array} \quad \begin{array}{c} f: \mu \rightarrow \mu: \text{idempotent} \\ A \oplus B \xrightarrow{\pi_1} B \xrightarrow{i_1} A \oplus B \end{array}$$

$$\tau_i^n: [n] \xrightarrow{\nabla_i^{n-1}} [n-1] \xrightarrow{\partial_i^n} [n] \quad (0 \leq i < n) \quad \text{"n-1" call}$$

is idempotent 2^n , induce $\mathbb{Z}^n$

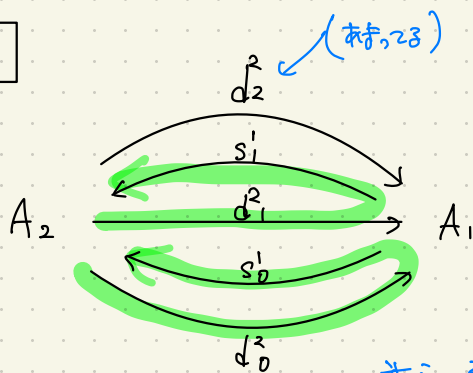
$$A_n \xrightarrow{\tau_i^n} A_n: \text{idempotent is.}$$

$$A_n \subseteq \text{Ker}(\tau_i^n) = \text{Ker}(s_i^n) \subseteq$$

$$\text{Im}(\tau_i^n) = \text{Im}(s_i^{n-1}) \wedge \in \text{direct sum} \text{ of } \mathbb{Z}^n.$$

$\Rightarrow \exists \text{ } \forall n \geq 0 \leq i < n \quad (2^{\mathbb{Z}} \subseteq \mathbb{Z}) \text{ } \overline{k} \in \mathbb{Z}$.

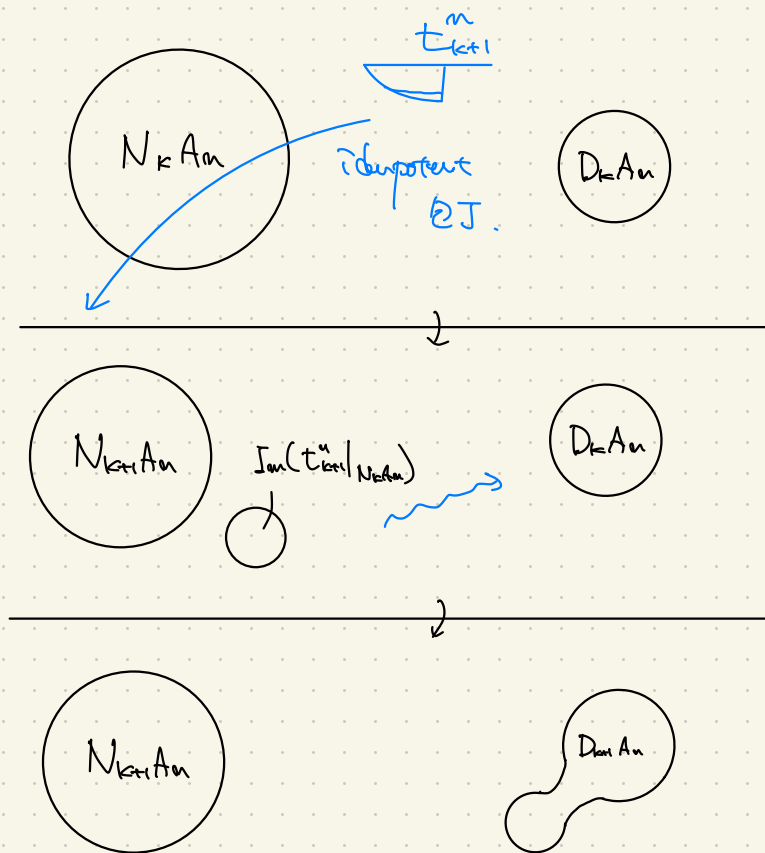
図 1



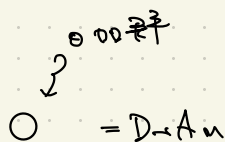
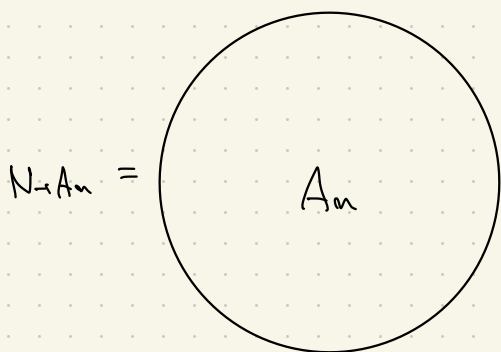
$\leftarrow E \cap \mathbb{R}^2$
 $\rightsquigarrow \{0 \in \mathbb{R}^2 \mid \mathbb{R}^2\}$
 $\text{射} \in \text{identity} \Rightarrow \mathbb{R}^2$
 $\therefore \text{simplicial identity}$

$\rightarrow * \tau_j^n \in \tau_j^n : A_n \longrightarrow A_n \in \mathbb{R}^c$
 $\leftarrow \tau_j^n \text{ 面} \in \mathbb{R}^c$

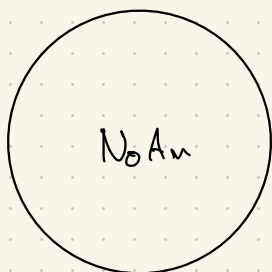
図 2



$\Rightarrow d \in \mathcal{C}_1 \text{ p2L.}$



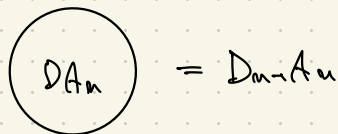
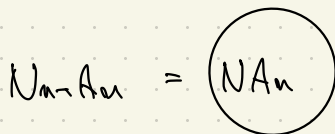
} t_0^n 移植



} t_1^n 移植

⋮

} t_{m-2}^n 移植



へ到達可能.

$$-1 \leq k < n \quad (2.0.12)$$

$$N_k A_n := \bigcap_{i=0}^k \text{Ker}(d_i^u) \quad (= \bigcap_{i=0}^k \text{Ker}(\tau_i^u))$$

$$D_k A_n := \sum_{i=0}^k \text{Im}(s_i^{u-1}) \quad (= \sum_{i=0}^k \text{Im}(\tau_i^u))$$

2.7.3.

$$\left(\begin{array}{l} A_n = N_{-1} A_n \supseteq \cdots \supseteq N_{n-1} A_n = N A_n \\ 0 = D_{-1} A_n \subseteq \cdots \subseteq D_{n-1} A_n = D A_n \end{array} \right) \quad (2.7.3)$$

$$A_n = N_k A_n \oplus D_k A_n \quad \forall \quad -1 \leq k < n \quad (2.7.4)$$

inductive 2.7.4. $\leftarrow$ 2.7.4. $k = n-1$ 2.7.4

$k = -1$ 2.7.4. easy.

$$k \rightsquigarrow k+1 \quad (k+1 < n)$$

2.7.4. τ_{k+1}^u は $N_k A_n$ 上 $\mathbb{A}^u$. (i.e. $\tau_{k+1}^u(N_k A_n) \subseteq N_k A_n$)

$$(\because \tau_{k+1}^u \circ \tau_i^u = \tau_i^u \circ \tau_k^u \quad (0 \leq i \leq k))$$

2.7.4. $N_k A_n$ は $\text{Ker}(\tau_{k+1}^u|_{N_k A_n}) \oplus \text{Im}(\tau_{k+1}^u|_{N_k A_n})$

1. 2.7.4 分解された.

$$= 2. \text{Ker}(\tau_{k+1}^u|_{N_k A_n}) = N_k A_n \cap \text{Ker}(d_{k+1}^u)$$

$$= N_{k+1} A_n \text{ of } n^2,$$

$$A_n = N_k A_n \oplus D_k A_n$$

$$= \underline{N_{k+1} A_n \oplus \operatorname{Im}(t_{k+1}^n |_{N_k A_n})} \oplus D_k A_n$$

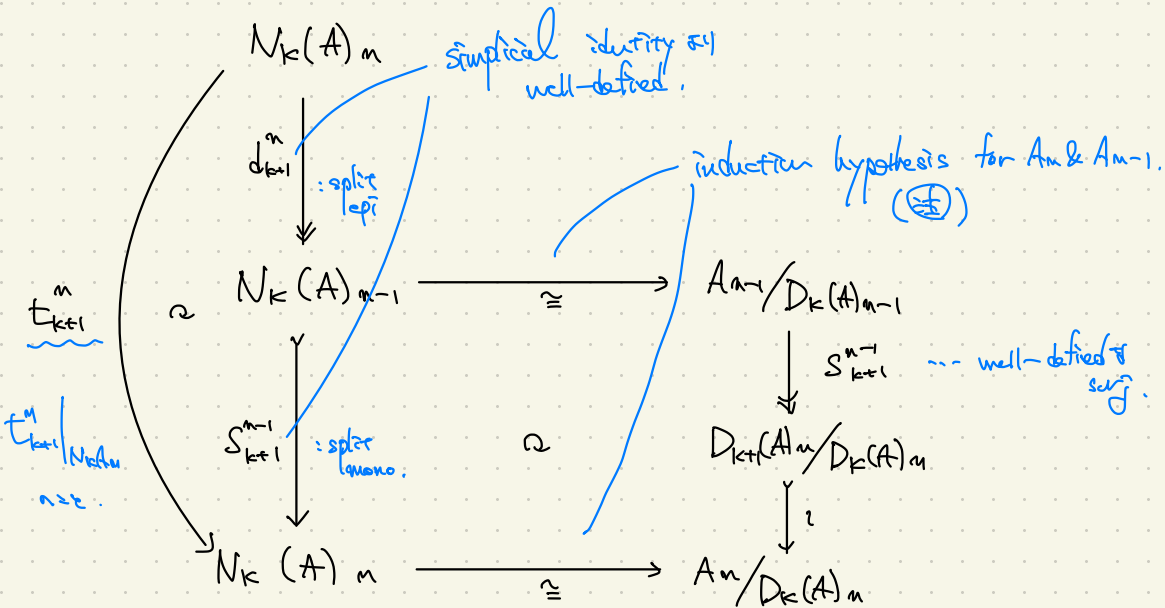
∴ 直和分解を得る.

$$\text{後は, } \operatorname{Im}(t_{k+1}^n |_{N_k A_n}) \oplus D_k A_n = D_{k+1} A_n$$

を示せばよい.

両辺, $D_k A_n \subseteq$ 全く同じ A_n の submodule of n^2 ,

$A_n / D_k A_n$ 中で考えるのはよい.



∴ 2.3. ∴ a diagram in image of $D_{k+1}(A_n/D_k(A)_n)$ is $k+1, 0 \leq k$

$$C = N \oplus D \quad \text{is} \quad N \cong C/D \quad \text{is simply} \quad \subset$$

$$N \cong \bar{\partial} + \frac{\partial}{\partial \bar{\partial}} \in \mathbb{C}^2 \text{ is } \mathbb{C}^2 \text{ is } \mathbb{C}^2!$$

$$\text{e.g.) } \text{Set}^{\Delta^{\text{op}}} \xrightarrow{F^{\Delta^{\text{op}}}} \text{Mod}_R^{\Delta^{\text{op}}} \quad \text{is}$$

$$X: \text{simplicial set} \cong \mathbb{Z}^{\mathbb{N}}, \quad \text{is} \quad FX \quad \cong \mathbb{Z}^{\mathbb{N} \times \mathbb{N}},$$

$$DFX_n = \bigoplus_{\substack{x \in X_n \\ \text{is degenerate}}} R_x \quad \text{is} \quad \mathbb{Z}^{\mathbb{N}},$$

"N is non-deg part"
 $\hookrightarrow$ a $\mathbb{Z}^{\mathbb{N} \times \mathbb{N}} \cong \mathbb{Z}^{\mathbb{N} \times \mathbb{N}}$
 $\hookrightarrow \mathbb{Z}^{\mathbb{N} \times \mathbb{N}} \cong \mathbb{Z}^{\mathbb{N} \times \mathbb{N}}$

$$NFX_n \cong CFX_n / DFX_n \cong \bigoplus_{\substack{x \in X_n \\ \text{is non-deg}}} R_x \quad \text{is} \quad \mathbb{Z}^{\mathbb{N} \times \mathbb{N}}!$$

$$T_2 \cong \mathbb{Z}^{\mathbb{N} \times \mathbb{N}} \quad \text{interval } \Delta_1 \quad \text{is } \mathbb{Z}^{\mathbb{N} \times \mathbb{N}}.$$

$$NFA_1 \quad \text{is}$$

$$\dots \xrightarrow{2} 0 \xrightarrow{1} R \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} R^2$$

is $\mathbb{Z}^{\mathbb{N} \times \mathbb{N}}$. standard interval
 object in Ch
 $\text{is } \mathbb{Z}^{\mathbb{N} \times \mathbb{N}}.$

§ 2.3 as a coreflector of $Ch_2 \rightarrow Mod_R^{\Delta^{op} mono}$

$\simeq \mathbb{Z}^2$. N : normalized chain complex

お程度 等当 Def $\mathbb{Z}^2$ 思ひ $\mathbb{Z}^2$ 17 $\mathbb{Z}^2$ (?) $\mathbb{Z}^2$ $\mathbb{Z}^2$,

function N お程度自然子 導出 $\mathbb{Z}^2$ お $\mathbb{Z}^2$.

実は. construction N は. (adjunction 意味)

chain complex $\simeq$ "semi-simplicial object" $\simeq$

$\mathbb{Z}^2$ $\mathbb{Z}^2$ 表裏一体 $\mathbb{Z}^2$ 子!

Recall (semi-simplicial object)

Δ_{mono} $\simeq$ Δ adj monic $\mathbb{Z}^2$ $\mathbb{Z}^2$ subcat $\mathbb{Z}^2$.

Δ_{mono} $\mathbb{Z}^2$ $\mathbb{Z}^2$ $\mathbb{Z}^2$ contravariant functor $\mathbb{Z}^2$.

$\mathbb{Z}^2$ semi-simplicial object. $\mathbb{Z}^2$.

Rank

$N: Mod_R^{\Delta^{op}} \longrightarrow Ch_2(Mod_R)$ は.

d_n "action construction". $Mod^{\Delta^{op}} \xrightarrow{res} Mod^{\Delta^{op} mono}$

$\mathbb{Z}^2$ $\mathbb{Z}^2$ は $\mathbb{Z}^2$. N は semi-simplicial module $\mathbb{Z}^2$

a construction $\mathbb{Z}^2$ $\mathbb{Z}^2$ $\mathbb{Z}^2$ $\mathbb{Z}^2$ $\mathbb{Z}^2$ $\mathbb{Z}^2$.

重要な observation は.

chain complex は. 特別の semi-simplicial module である
 $= e.$

chain complex (C, ∂) は. (convention. ~~通常~~ (12) への注意. 今 C_n は n 次元.)

$$d_i^n = \begin{cases} (-1)^n \partial_n & (i=n) \\ 0 & (\text{other}) \end{cases}$$

$e \in C_n = e \in C^n$. semi-simplicial module $C \in \mathcal{F}^n$.

5.41 は.

$$d_i^n = 0 \quad (0 \leq i < n)$$

半単純 semi-simplicial module は chain complex である.

chain complex である.

$$d_i^n = 0 \quad (0 \leq i < n) \quad \text{かつ} \quad \dots \quad \textcircled{*}$$

semi-simplicial module である

$$\left(N_{A_n} := \bigcap_{0 \leq i < n} \text{Ker}(d_i^n) \quad \text{連続性がある} \right)$$

সিদ্ধান্ত

$$\text{Ch}_2(\text{Mod}_R) \xrightleftharpoons[\text{N}]{\text{I}} \text{Mod}_R^{\Delta_{\text{mono}}^{\text{op}}}$$

is coreflective sub.

Remark

$$\text{Ch}_2(\text{Mod}_R) \xrightleftharpoons[\text{Mod}^2]{\text{Af-Mod}^2} \text{Mod}_R^{\text{Mod}^2 \text{ op}} \quad \text{এই মাপকাঠি}$$

bi-reflective sub. $\in \mathcal{A}$. lim, colim $\in \mathcal{A}$ $\Rightarrow$ $\mathcal{A}$ -ইয়ে।

$\Leftarrow$ Af-enriched $\mathcal{A}$ lex op functor α $\Rightarrow$ $\mathcal{A}$ -ইয়ে।

§ 3. Dold - Kan Correspondence

7.12. Dold - Kan Correspondence 証明.

§ 3.1 semi-simplicial $\rightsquigarrow$ simplicial

まず, $Ch_2(Mod_R) \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\perp} \\ \downarrow \end{matrix} Mod^{\Delta^{op}}_{mono} \cong \text{etc.}$

よって $Mod^{\Delta^{op}}_{mono} \subset Mod^{\Delta^{op}} \cong \text{etc.}$

= 28 Categorical $\cong$ $\neq$ $\subset$ $\neq$ $\cong$ は、当然

1. relation $\cong \Rightarrow$ (Kan extension!)

Prop (simplicial $\mathcal{C}$: degenerate n - $\mathbb{Z}$ 式 $\mathbb{Z}$ $\mathbb{Z}$)

$\mathcal{C}$: cocomplete category $\subset \mathbb{Z}$ $\mathbb{Z}$.

$\mathcal{C}^{\Delta^{op}}_{mono} \xleftarrow{\text{res}} \mathcal{C}^{\Delta^{op}}$ restriction if left adjoint

$\mathcal{C}^{\Delta^{op}}_{mono} \xleftarrow{\text{res}} \mathcal{C}^{\Delta^{op}} \cong \text{etc.}$

$A_n \in \mathbb{Z}$, $\mathbb{Z}$ is "2" $\neq 0$.

$(\mathbb{Z}A)_n = \coprod_{[n] \xrightarrow{\mathbb{Z}} [n]} A_n$
 epic

e.g.) $\text{Mod}_R \xrightarrow{\Delta^{\text{op}}} \text{Mod}_R^{\Delta^{\text{op}}}$

z'rā. $\text{PA}_m = \bigoplus_{[n] \xrightarrow{\varphi} [m]} A_n$ ✓✓✓.

$\text{DPA}_m = \bigoplus_{\substack{[n] \xrightarrow{\varphi} [m] \\ \text{not id}}} A_n$ ✓✓✓, 旋る?

$\text{NPA}_m \cong A_m$ ✓✓✓ (お! ✓✓✓)

★ ★ ★. A : simplicial module 12su217.

NA is semi-simplicial 12173.

$$\leadsto \left(\begin{array}{l} X \xrightarrow{f} Y = \text{subtraction } \neq, \\ fX \longrightarrow fY \\ x \longmapsto \begin{cases} f(x) & (\text{if } x \in \text{dom } f) \\ 0 & (\text{other}) \end{cases} \\ 0 \text{ is } \text{ob} \text{ } \lambda \text{ is } \text{e} \text{ } \theta^{\text{an}} \text{ } \text{is} \text{ } \text{is} \text{ } \end{array} \right. \quad \text{e} \text{ } \text{is} \text{ } \neq \text{ } \text{is} \text{ } .$$

§ 3.2 Proof of Dold - Kan Correspondence.

equiv is, Abelian cat. 2-in-3 cat

Thm (Dold - Kan Correspondence)

R = ring (2-in-3).

$$\text{Ch}_{\geq}(\text{Mod}_R) \begin{matrix} \xrightarrow{I} \\ \xleftarrow[N]{\perp} \end{matrix} \text{Mod}_R \begin{matrix} \xrightarrow[\Delta_{\text{mono}}]{\text{op}} \\ \xleftarrow[\text{res}]{\perp} \end{matrix} \text{Mod}_R^{\Delta^{\text{op}}}$$

or 合成 a adjunction is.

(adjoint) equivalence $\exists$ 5.23.

(proof) adjoint equivalence of 2, with counit 0
is of 2 in 3.

unit

本質は $\mathbb{P} \subset C$. 以下, I & res は 同値.

$C = (C, d)$: chain complex (2-in-3).

$$\mathbb{P}C_n = \bigoplus_{\substack{0 \leq i \leq n \\ d^i \neq 0}} C_n \quad 2\text{-in-3}.$$

$$N\mathbb{P}C_n = C_n (\subseteq \mathbb{P}C_n) \text{ with.}$$

$$\text{with } C_n \xrightarrow{(\eta)_n} N\mathbb{P}C_n \text{ is Iso.}$$

($\leftarrow \eta_n$ is not: 一般論より, $C \xrightarrow{\eta^N} NC \xrightarrow{N\eta^P} N\mathbb{P}C$
 2-in-3 cat. 2-in-3 cat. nature of the equivalence.)

count

二つの変数。

$A = \text{simplicial } R\text{-module}$.

$E_A: PNA \longrightarrow A$ は.

$$(E_A)_n: \bigoplus_{[n] \xrightarrow{\Gamma} [n]} NA_n \longrightarrow A$$

$$\begin{array}{ccc} \psi & & \psi \\ x^\Gamma & \longmapsto & x * \Gamma \end{array}$$

x^Γ は x の Γ 部分.

これは iso であることが示される.

これは simplicial set X 上の x は $x = y * \Gamma$ である.
 $\uparrow$ non-deg $\uparrow \varphi$
 unique であることは証明されている!

$(E_A)_n$ は iso である.
 (これは NA_n である)
 NE_A は iso である, したがって NE_A は A の同型写像である.

これは集合である.

claim

$$N: \text{Mod}_R^{\Delta^{\text{op}}} \longrightarrow \text{Ch}_2(\text{Mod}_R) : \text{conservative}$$

monotonicity

N-equiv for Δ^{op}

claim n proof $\{2, 1, 0\} \xrightarrow{\Delta^{\text{op}}} \{2, 1, 0\}$

$$\Delta \xrightarrow{\text{succ}} \Delta$$

$$\downarrow$$

$$[n]$$

$$[n+1]$$

$$\ni i$$

$$f \downarrow$$

$$\mapsto$$

$$\downarrow f+1$$

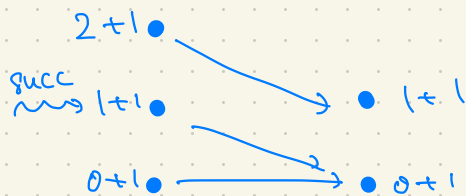
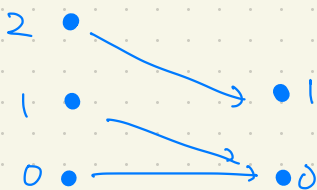
$$[m]$$

$$[m+1]$$

$$\ni \cdot$$

$$= \begin{cases} f(j)+1 & \text{if } i=j+1 \\ 0 & \text{if } i=0 \end{cases}$$

def Δ .



$$0 \left[\cdot \right] \longrightarrow \cdot \quad \text{not a map}$$

Δ is a (pointed) endofunctor succ is a natural transformation.

$$\begin{array}{ccc} \Delta & \xrightarrow{\text{id}_\Delta} & \Delta \\ & \Downarrow \text{succ} & \\ \Delta & \xrightarrow{\text{succ}} & \Delta \end{array}$$

Natural!!

存在 $\Rightarrow$ 12 induce Σ 12 / "Path Space" 3cui?

$$p := - \circ \text{succ}^p$$

$$\text{Mod}_R^{\Delta^{\text{op}}} \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow d_0 \\ \xrightarrow{\quad} \end{array} \text{Mod}_R^{\Delta^{\text{op}}} \quad \text{Id} \quad \text{or } \text{Id}$$

具体的には.

A : simplicial module 12 12. $(PA)_n = A_{n+1} \Sigma^n$.

$PA \xrightarrow{d_0} A$ n n -index component 12

$A_{n+1} \xrightarrow{d_0^{n+1}} A_n$ Σ^n 12.

d_0 n Kernel Σ $\Lambda \in \mathcal{P}$. ("Loop Space" 3cui?)

$0 \longrightarrow \Lambda \longrightarrow P \longrightarrow \text{Id} \longrightarrow 0$ Σ^n exact sequence.

12 12. A : simplicial module n n -index 12 12. 12

$$0 \longrightarrow \text{Ker}(d_0^{n+1}) \xrightarrow{\text{inclusion}} A_{n+1} \xrightarrow{d_0^{n+1}} A_n \longrightarrow 0$$

Σ^n exact sequence 12.

($A_n \in A_{n+1} \Sigma^n$ 12 12, induction 12 12 12 12)

Λ $\mathcal{P}$ 12. (12 $n = k$ $n+1 = k$ Σ^n 12 12 12 12)

$N \in$ 12 12. (proof n 12 12) (12. $\text{Ker}(d_0^{n+1})$ 12 12.)

(proof of claim)

$$0 \leq k \text{ is an integer.}$$

Naive (2.12). Grothendieck: inductive.
 (claim is ZFC + CH (2.12))
 (2.12 is a ZFC + CH...)

$$\left| \begin{array}{l} \text{claim}^k \text{ is true } \forall f: A \longrightarrow B \text{ in } \text{Mod}_R^{\Delta^{\text{op}}} \text{ is true,} \\ Nf: \text{iso} \implies f_n: \text{iso} \end{array} \right.$$

$\exists$ inductive integer.

$$\frac{\text{claim}^0}{(Nf)_0 = f_0 \text{ is ok.}}$$

$$\frac{\text{claim}^k \implies \text{claim}^{k+1}}{\text{claim}^k \text{ is true.}}$$

claim^k is true.

$$f: A \longrightarrow B \text{ in } \text{Mod}_R^{\Delta^{\text{op}}}$$

$$\text{if } Nf: \text{iso} \text{ then } f \text{ is iso.}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta A_k & \longrightarrow & \overset{A_{k+1}}{P A_k} & \twoheadrightarrow & A_k \longrightarrow 0 \\ & & \downarrow \Delta f_k & & \downarrow P f_k = f_{k+1} & & \downarrow f_k \\ 0 & \longrightarrow & \Delta B_k & \longrightarrow & \underset{B_{k+1}}{P B_k} & \twoheadrightarrow & B_k \longrightarrow 0 \end{array}$$

is five lemma $\exists$ $\frac{1}{\text{is}} \text{ is } \text{is}$

つまり, Δf_k と f_k が iso だと示せばいい.

f_k : iso だと claim^k (induction hypothesis) により ok.

Δf_k : iso だと示す.

よって, claim^k を使えば, (NAf) : iso だと示せばいい.

$$\Rightarrow 2^{\circ}, (NAA)_m = \underbrace{\text{Ker}(d_0^{n+1})}_{\substack{\parallel \\ AA_m}} \cap \bigcap_{i=0}^{m-1} \underbrace{\text{Ker}(d_{i+1}^{n+1})}_{\substack{\parallel \\ P(\partial_i^n)}}$$

$$= (NA)_{n+1} \subseteq PA_m$$

$$A \text{ の } (NAB)_m = (NB)_{n+1} \text{ だと示す.}$$

$$(NAA)_m \xrightarrow{NAf_m} (NAB)_m$$

$$\parallel \quad \parallel$$

$$NA_{n+1} \xrightarrow[\substack{\parallel \\ (Pf)_n / AA_m / NAA_m}]{Nf_{n+1}} NB_{n+1}$$

つまり, 仮定 Nf : iso

$$\text{よって } NAf : \text{iso}, \quad \left(\begin{array}{l} \hookrightarrow (Af)_k : \text{iso} \text{ だと five lemma により} \\ f_{k+1} : \text{iso} \end{array} \right)$$

$\Rightarrow 2^{\circ}$, Dold-Kan correspondence により.

$\frac{1}{B} \cong A \oplus B$ により. どちらも.