

§1. transfinite composite of pointed endofunctor.

§2. 合成則の停留値 $\in$ fixed object.

§3. $\text{Fix}_e T \xrightarrow{\perp} \mathcal{C}$

§1. transfinite composite of pointed endofunctor.

■ Ord は. ordinal $\omega \in$ 1273 poset (proper class 2215...)

■ ω と $\omega+1$ は ω の ω 倍.

本当は、 ω は ω 倍の ω 倍... (2215 2215...)

Def (transfinite composite of pointed endofunctor)

$\mathcal{C}$: cocomplete category.

$I_{\mathcal{C}} \xrightarrow{0} T$ in $\mathcal{C}^{\mathcal{C}}$ (pointed endofunctor) 1273 1273,

$\text{Ord} \xrightarrow{T^{(-)}} \mathcal{C}^{\mathcal{C}} \ni T^{\alpha}$, α は transfinite ordinal

2215 2215,

• $\alpha = 0$ or ω

• α : ordinal 1273 1273.

• α : limit ordinal 1273 1273.

$T^{\alpha} = T^0 := id_{\mathcal{C}}$

$T^{\alpha} \xrightarrow{T^{\alpha \leq \alpha+1}} T^{\alpha+1} \in \mathcal{C}^{\mathcal{C}} \ni T^{\alpha} \in \frac{1}{2} \times \frac{1}{2}$.

$T^{\alpha} := \text{colim}_{\beta < \alpha} T^{\beta} \in \frac{1}{2} \times \frac{1}{2}$.

(1273 2215)

Def (停留)

$\mathcal{C}$: cocomplete. $\text{id}_{\mathcal{C}} \xRightarrow{0} T$ 12.2.12.

$A \in \text{ob } \mathcal{C}$ 合成列 $T^{\ell}(A) = \text{Ord} \longrightarrow \mathcal{C}$

Def α : ordinal $\mathcal{C}$ 停留可成列. $\mathcal{C}$ 上の同値列の条件

$\alpha \mapsto (\text{従, 2 全 2})$ $\mathcal{C}$ 三項可成列 $\mathcal{C}$ 可成.

(i) $\alpha \leq \beta \leq \gamma$ $T^{(\beta \leq \gamma)}(A) : T^{\beta}A \longrightarrow T^{\gamma}A : \text{iso}$

(ii) $T^{(\alpha \leq \alpha+1)}(A) : T^{\alpha}A \longrightarrow T^{\alpha+1}A : \text{iso}$

(iii) $T^{\alpha}(A) : \text{fixed object.}$

(proof of (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii))

(i) $\Rightarrow$ (ii) : $\beta = \alpha$. $\gamma = \alpha + 1$ $\mathcal{C}$ 可成列 $\mathcal{C}$.

(ii) $\Rightarrow$ (iii) : $T^{(\alpha \leq \alpha+1)}(A) = \mathcal{O}_{T^{\alpha}(A)}$ $\mathcal{C}$.

(iii) $\Rightarrow$ (i) : $\mathcal{C}$ 可成列, $\beta = \alpha$ $\alpha \in \mathbb{N}$, $\gamma \geq \alpha$ transfinite induction

$\mathcal{C}$ 可成列. $\gamma \in \mathbb{N}$. $\forall \beta \geq \alpha$. $T^{\beta}(A) : \text{fixed object } \mathcal{C}$ $\mathcal{C}$.

$\beta \geq \alpha \in \text{fix } \mathcal{C}$. $\forall \gamma \geq \beta$ $\mathcal{C}$ transfinite induction $\mathcal{C}$ 可成.

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Def (τ_A)

$\mathcal{C}$: cocomplete. $\text{Id}_{\mathcal{C}} \xrightarrow{0} T$ 12.24.2.

- $A \in \text{ob}$ とは, $\exists \alpha: \text{ordinal}, A \text{ の合成列は } \alpha \text{ で停留する}$ とき, 停留する合成列 がある. $\hookrightarrow$

- $A \in \text{ob } \mathcal{O}''$ 停留可能な合成列に属するとき,

$$\tau_A := \min \{ \alpha \text{-ordinal} \mid A \text{ 合於 } \alpha \text{ 之停留点} \}$$

- $\forall A \in \text{ob } \mathcal{C}$, A は停留する合成列 $\exists$ といふ.
 T は停留する $\exists$ といふ.

Rank T は停留 ∞ かつ $\{ \tau_A \mid A \in \text{obj } \mathcal{C} \}$ は Ord の非昇列でない

Remark τ_A is "ordinal invariant" is it not? τ_A is not invariant under ordinal isomorphism.

<https://mathoverflow.net/questions/11191/nonhausdorff-dimension>

$$2^{\text{nd}} \text{ } f''(2) = 12.$$

Top f. $\overline{\Delta}_X \hookrightarrow X \times X \xrightarrow[\pi_2]{\pi_1} X \xrightarrow{\theta_X} TX : \text{regularizer}$

$$Z^1 \cong \text{id} \xrightarrow{\theta} T \text{ なる } \tau_A \text{ なる } \tau_A \text{ なる } \tau_A$$

Def (T^{Ord})

$\mathcal{C}$: cocomplete. $Id_{\mathcal{C}} \xRightarrow{0} T$ 12242.

T is $\frac{1}{2}$ $\exists \exists \exists \exists$, $T^{Ord} : \mathcal{C} \longrightarrow \mathcal{C}$ 12242.

object: $T^{Ord}(A) := T^{\tau_A}(A)$

morphism: $A \xrightarrow{f} B$ 12242. $\tau_A, \tau_B \leq \gamma$ $\exists \exists \gamma$: ordinal

$$\begin{array}{ccc} \exists \exists \exists \exists, & T^{Ord} A = T^{\tau_A}(A) & \xrightarrow[\cong]{T^{\tau_A \leq \gamma} A} T^{\gamma} A \\ \downarrow \exists! & \downarrow T^{Ord} f & \downarrow T^{\gamma} f \\ & T^{Ord} B = T^{\tau_B} & \xrightarrow[\cong]{T^{\tau_B \leq \gamma}} T^{\gamma} B \end{array}$$

$\exists \exists \exists \exists$.

(check: γ is $\exists \exists \exists \exists$)

Remark = 4.12. $\text{colim}_\alpha T^\alpha$ is "large colimit" $\in \mathbb{R} \cdot \mathbb{Z}$.

$$\begin{array}{ccc} & \exists! & \\ & \nearrow T^{Ord} & \\ \mathcal{C} & & \text{Fix}_T \\ & \searrow \cong & \\ & \xrightarrow{T^{Ord}} & \mathcal{C} \end{array}$$

$\exists$. T^{Ord} is Fix_T is a functor $\wedge \text{is } \exists \exists$.

§ 3. $\text{Fix}_e T \xrightarrow{\perp} \mathcal{C}$

7th 12. Coobject $\mathbb{Z}$ Fixed object $12 \leq 3$.

Q. 2.12.10 inclusion a left cofinite set?

条件無しに $\exists$ universal 証明

Lemma (T^{OSA} A a 33.1 universal)

$\mathcal{C}$: cocomplete. $\text{Id}_{\mathcal{C}} \xRightarrow{\theta} T, \quad \varepsilon \in \mathcal{A}.$

⇐ 즉, 반대 $A \in \text{ob } \mathcal{C}, B \in \text{ob } \text{Fix}_{\mathcal{C}} T, A \xrightarrow{f} B$ 이 주어지면

A 合成 $3/1/3$ B 合成 $2/2/2$ $\varphi: T^{(1)}A \Rightarrow B$ 2- π , 2

$\gamma_0 = f$ 以外は γ 存在する. (unique existance.)

(proof) transfinite recursion 2.3.

$$\varphi_0 := f \in C,$$
 α is $\alpha+1$ in step 2's.

$$\begin{array}{ccc} T^{\alpha} A & \xrightarrow{\Theta^{\alpha, \alpha}} & T^{\alpha+1} A \\ \varphi_{\alpha} \downarrow & \alpha & \downarrow T\varphi_{\alpha} \\ B & \xrightarrow[\cong]{\Theta_B} & TB \end{array} \quad (\Theta \text{ a natural isomorphism})$$

$$\text{d) } \varphi_{\alpha+1} := \Theta_B^{-1} \circ T\varphi_\alpha$$

α : limit ordinal iff $\text{cultivate a universality } \exists (\beta)$.

Rank = $\alpha \leq \beta, \beta < \gamma \Rightarrow \forall \alpha = \text{ordinal}$. $A \xrightarrow{T^{\alpha \leq \beta}} TA \xrightarrow{\eta} \downarrow \varphi_\alpha \quad \#3 = \text{rank of } \eta \text{ is } \beta$.

Lemma ($\theta: \text{epic} \Rightarrow T^{\alpha \leq \beta} A = \text{epic}$)

$\mathcal{C}$: cocomplete. $\text{id}_{\mathcal{C}} \xrightarrow{\theta} T: \underline{\text{epic}} \quad \#3.$

$\exists R \text{ on } A \text{ s.t. } \exists \text{ ordinal } T^{\gamma} A = \text{Ord} \longrightarrow \mathcal{C}$

"image is $T^{\gamma} A$ epic

(proof) $\theta: \text{epic}$. $\mathcal{C}$: cocomplete $\#2$. θ on components $\neq \text{epic}$ $\#3$.
 $\#4$ is, transfinite induction. (limit step is, cocomplete $\#3$ is $\#2$) $\#4$

Thm ($\text{Fix}_{\mathcal{C}} T \xrightarrow{\perp} \mathcal{C}$)

$\mathcal{C}$: cocomplete. $\text{id}_{\mathcal{C}} \xrightarrow{\theta} T: \underline{\text{epic}} \quad \#3.$

T^{Ord} 停留場 到達せず,

$\text{Fix}_{\mathcal{C}} T \xleftarrow{T^{\text{Ord}}} \mathcal{C} \xrightarrow{\text{inclusion}} \mathcal{C} \quad \#1 \text{ 成立.}$

unit is $A = T^0 A \xrightarrow{T^{\text{Ord}}} T^{\infty} A = T^{\text{Ord}} A$

is $\#3$ is $\#3$.

(proof) $\forall A \supseteq \text{a Lemma}$ を用いるわけ, $T^{\alpha \leq \tau_A} A$ だ

$\text{Fix}_e T \hookrightarrow C$ a A に対する universal arrow
 があることを示す //

停留可能な T については いさゝか分るが、

停留をどう判定するか?

$\leadsto$ いくつかの条件がある。

($\forall A \in \text{ob } C, A/e$ 上の epic 射は
 full subcategory a skeleton だ。
 small だ)

Prop (停留の便利な条件)

C : cocomplete, co-well-powered, $\text{id}_e \xrightarrow{\theta} T$: epic だ。

T は 停留可能 (後, 2 Lemma 使った)

(proof) 仮に $T^{\alpha} A \in \text{ob } C$ だと停留 出来る。

と仮定, $\forall \alpha$ ordinal. $T^{\alpha \leq \alpha+1} A$ is not iso.

$T^{\alpha} A : \text{Ord} \longrightarrow C$ は A の co-Subobject だ

small poset $\text{coSub}_e(A)$ を経由する,

$\{T^{\alpha} A\}_{\alpha \in \text{Ord}}$ への $\text{coSub}_e(A)$ に対する proper size の降下列

存在 $\mathcal{C}$. $\mathcal{C} = \text{co-well-powered}$ 12 矛盾 $\rightarrow$

e.g.) $\mathcal{C} = \text{Top}$ 12 矛盾. $\bar{\Delta}_X \xrightarrow{\gamma} X \times X \xrightarrow[\pi_2]{\pi_1} X \xrightarrow{\theta_X} TX$
 $\mathcal{C}$ 12 $\text{id}_{\text{Top}} \xrightarrow{\theta_X} T \neq \text{id}_T$.

Top 12 cocomplete & co-well-powered 12.

$\text{Fix}_\mathcal{C} T \xhookrightarrow{\perp} \mathcal{C} \neq \emptyset$.

$X = \text{fixed object} \Leftrightarrow \theta_X = \text{id}_X$

$\Leftrightarrow \bar{\Delta}_X \xrightarrow{\gamma} X \times X \xrightarrow[\pi_2]{\pi_1} X \xrightarrow{\gamma} X$

$\Leftrightarrow \bar{\Delta}_X \subseteq \Delta_X$

$\Leftrightarrow X: \text{Hausdorff}$

2nd. T^{ord} 12 largest Hausdorff quotient $\neq \emptyset$!

Remark dual $\neq \mathcal{H}(\mathcal{C})$. well-powered & complete category $\mathcal{C}$ 12 12. T 12 subfunctor T 12 12 同不変の adjunction 12 12!

($\rightarrow$ 12 12 12 12 12 12)

e.g.) $\mathcal{C} = \text{Group}$. $T := [-, -] = \text{交換子群}$.

$\in \mathcal{B}$. $\Rightarrow n \in \mathbb{N}$. $\text{Fix}_e T$ is perfect group and full subcategory.

$$\begin{array}{ccc} \text{Perfect Group} & \xrightarrow{\text{inclusion}} & \text{Group} \\ & \downarrow \perp & \\ & \xleftarrow{[-, -]^{\text{ord}}} & \end{array}$$

$G := \text{可解} \Leftrightarrow T_G = \text{finite} \ \& \ [-, -]^{\text{ord}}(G) = \text{trivial}$

Remark

別々の $\mathcal{C}$ の adjunction α の η は 4 個ある.

T^{ord} α . $\mathcal{C} \subseteq \text{idempotent monad}$ である.

. $\therefore$ $\mathcal{C}$ にも $\mathcal{B}$ の $\mathcal{C}$ がある. $\Rightarrow$ $\mathcal{C}$ は $\mathcal{B}$ の $\mathcal{C}$ である.