

§ 10.5 Simplicial Homotopy

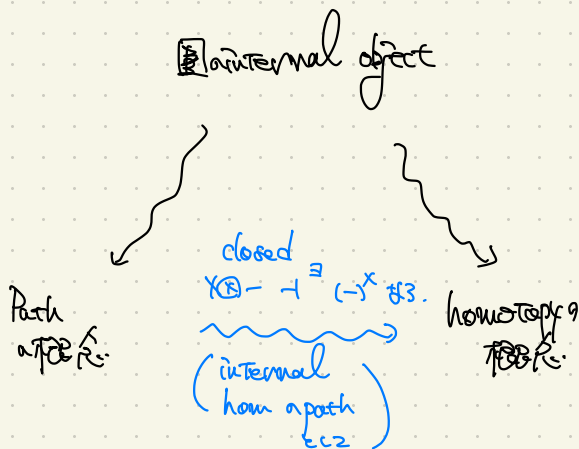
homotopy の概念を定義する.

実は, Categorical を視点から見る.

- 圏 $\mathcal{C}$ の射の間の homotopy は何か?
- 圏 $\mathcal{C}$ の "internal object" は何か?

は非常によく似た問題である.

そこで見てみる.



e.g.)

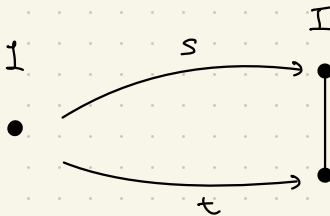
$\mathbb{R} \subseteq \mathbb{R}$	interval	Path	homotopy
Top	$[0, 1]$	$\gamma: [0, 1] \rightarrow X$ path	$\gamma: [0, 1] \rightarrow X$ homotopy
Cat	$\bullet \longrightarrow \bullet$	morphism	Natural Transformation
Cat	$\bullet \rightleftarrows \bullet$	isomorphism	Natural iso
$Ch(\mathcal{A})$	"standard interval"	$y_0, y_1 \in C_0 \cong \mathbb{Z}^n$ $dy_0 = y_1 - y_0 \neq 0$ $x \in C_1, dx = y_1 - y_0$	<u>chain homotopy</u>
Poset	$0 \leq 1$	$x \leq y \iff \exists f: x \rightarrow y$ a.k.a. $x \leq y$	(pointwise) $f \leq g$
Set Δ^p	Δ_1	$x_0 + x_1 + \dots + x_p = 1$	$x_0 + x_1 + \dots + x_p = 1$ (simplicial homotopy)
Manifold	$\mathbb{R}$	smooth path	smooth homotopy
Topos	?	?	?
Convex Space	$[0, 1]$	$\frac{1}{2}x + \frac{1}{2}y$	affine to $\frac{1}{2}x + \frac{1}{2}y$
$\frac{1}{2}x + \frac{1}{2}y$	$(0, 1), (0, 1)$	$(x, A) \sim (y, B)$ A and B paths	$(x, A) \xrightarrow{f} (y, B)$ is a homotopy $f: [0, 1] \rightarrow B$ is a path
Dir Graph	$\bullet \longrightarrow \bullet$	edge	$f(v) \rightarrow g(v) \neq 0$ $v \in E$ edge

Def (interval object 定義 $\mathbf{I}$)

monoidal category $(\mathcal{C}, I, \otimes, \alpha, \rho, \lambda)$ a interval object

interval object $I \in \mathcal{C}$, I has a 2 object $1 \xrightleftharpoons[t]{s} I$ の組

(I, s, t) である。



Def (Path)

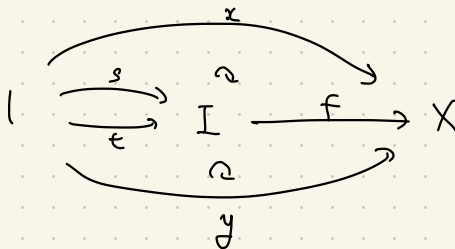
path (経路) は I の元

$\mathcal{C} = \text{monoidal}$. $I \xrightleftharpoons[t]{s} I$ interval である。

• $X \in \mathcal{C}$ a (1-) element である。 $1 \xrightarrow{x} X$ $a \in \mathcal{C}$.

• $X \in \mathcal{C}$ $1 \xrightarrow{x} X$ である。 $1 \xrightarrow{y} X$ である path である。

$I \xrightarrow{f} X$ である。 $f s = x$, $f t = y$



path $f \in \mathcal{C}$ である。

Def (homotopy)

$\mathcal{C}$: monoidal. $I \xrightleftharpoons[t]{s} I$: internal object.

in $\mathcal{C}$. $X \xrightleftharpoons[g]{f} Y$ in $\mathcal{C}$. $f \circ s \circ g$ a homotopy

is

$$X \otimes I \xrightarrow{H} Y \text{ is } \#12.$$

$$\begin{array}{ccccc} & & f & & \\ & \searrow & \cap & \searrow & \\ X \cong^{P_x} X \otimes I & \xrightarrow{id_X \otimes s} & X \otimes I & \xrightarrow{H} & Y \quad \dots \textcircled{*} \\ & \xleftarrow{id_X \otimes t} & \cap & \xleftarrow{g} & \\ & & g & & \end{array}$$

is $\#12$ in $\mathcal{C}$.

Remark (homotopy is internal hom a path)

$X \otimes -$ is right adjoint $[X, -]$ is $\#12$.

diagram $\textcircled{*}$ is transpose is.

$$\begin{array}{ccccc} & & [f] & & \\ & \searrow & \cap & \searrow & \\ I & \xrightarrow{s} & I & \xrightarrow{H^b} & [X, Y] \\ & \xleftarrow{t} & \cap & \xleftarrow{g} & \\ & & [g] & & \end{array}$$

is $\#12$.

$[X, Y]$ is $\#12$ is $[f]$ is $[g]$ is $\#12$ is $\#12$.

e.g.) $\mathbb{A}^1$ is interval object. $I \xrightleftharpoons[\text{id}]{\text{id}} I$

$\leadsto x = y$ is $\mathbb{A}^1$ path is equality.

$\leadsto X \xrightleftharpoons[g]{f} Y$ is $\mathbb{A}^1$ homotopy is equality.

e.g.) $I \xrightleftharpoons[i_2]{i_1} I \sqcup I$ is $\mathbb{A}^1$ on $\mathbb{A}^1$ ends.

$\leadsto x = y$ is $\mathbb{A}^1$ path is $\rightarrow$ and $\leftarrow$.

$\leadsto$ homotopy is, closed is $\mathbb{A}^1$.

e.g.) $\mathcal{C} = (\text{Top}, X, 1)$ is $I \xrightleftharpoons[\tau]{s} [0, 1]$ is $\mathbb{A}^1$.

$\leadsto$ path is $\mathbb{A}^1$ is $\mathbb{A}^1$.

$\leadsto$ homotopy is $\mathbb{A}^1$ is $\mathbb{A}^1$.

e.g.) I is $\mathbb{A}^1$ is $\mathbb{A}^1$.

Remark

Categorical is X is X .

• homotopy is $\mathbb{A}^1$ is $\mathbb{A}^1$ is $\mathbb{A}^1$.

• $\mathbb{A}^1$ functor is $\mathbb{A}^1$ is $\mathbb{A}^1$.

is $\mathbb{A}^1$.

Set Δ^{op} is. $X \in \Delta_0 \mathbb{Z}$ (Cartesian) monoidal.

$$[0] \xrightleftharpoons[d'_0]{d'_1} [1] \quad \text{in } \Delta \text{ induce } \mathbb{Z} \mathbb{Z}.$$

$$\Delta_0 \xrightleftharpoons[i_1]{i_0} \Delta_1 \quad \text{is } \mathbb{Z} \mathbb{Z}.$$

$\mathbb{Z} \mathbb{Z}$ interval $= \mathbb{Z} \mathbb{Z} = \mathbb{Z} \mathbb{Z} \mathbb{Z}$.

Path

simplicial set X $x, y \in X_0$

$\mathbb{Z} \mathbb{Z}$. $x \neq y$ a path is. $\ell \in X_1 \mathbb{Z}$, $\ell * d'_0 = x$

$\Delta_1 \xrightarrow{\ell} X$

$\ell * d'_1 = y$
 $\mathbb{Z} \mathbb{Z} \neq \mathbb{Z}$

$\Delta^{\text{op}} (\Delta_0, X)$

Def (Simplicial homotopy)

$\text{Set}^{\Delta^{\text{op}}}$ a cartesian monoidal structure $\times$.

$$\Delta_0 \xrightleftharpoons[i_1]{i_0} \Delta_1 \quad \#3 \text{ interval } \text{is } \Delta_1$$

homotopy $\simeq$ Simplicial homotopy $\simeq$. ↑ topological set

これに 書き下す Δ_0 今日 Δ_1 の Δ_0 である。

すなわち、 Δ_0 Δ_1 interval object $\Delta_0 \xrightleftharpoons[i_1]{i_0} \Delta_1$

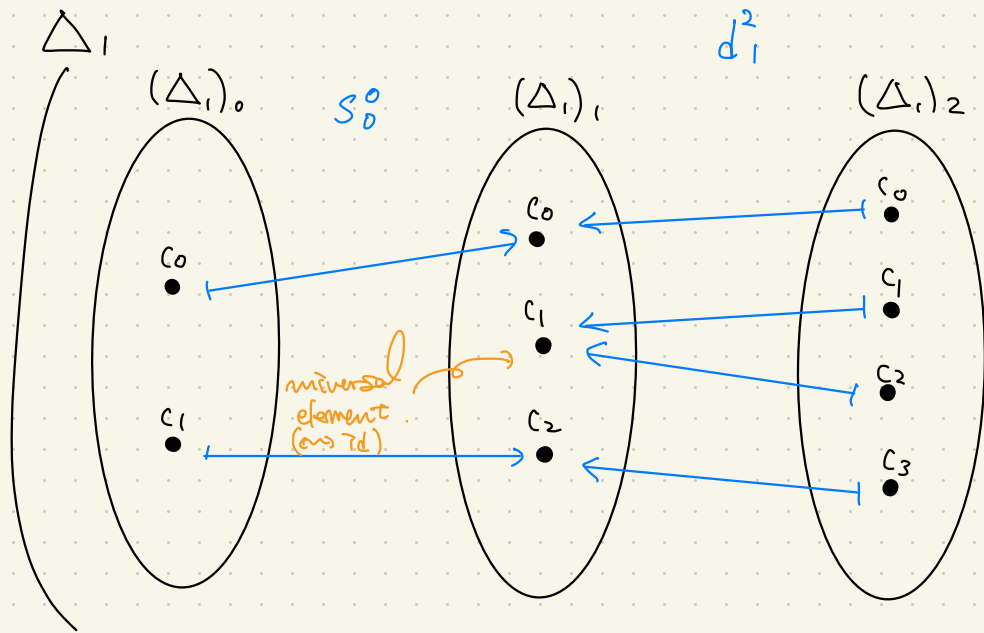
は Δ_0 Δ_1 である。

Δ_1 is Δ_1

$$(\Delta_1)_n = \Delta([n], [1]) \quad \text{is } n+2 \text{ elements.}$$

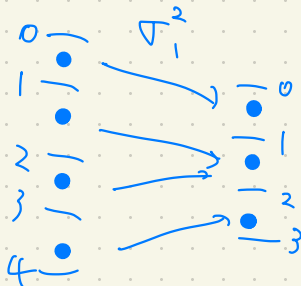
$$[n] \xrightarrow{C_i} [1] \quad (0 \leq i \leq n+1) \text{ is.}$$

$$C_i^{-1}(0) = \{0, \dots, i-1\} \quad \#3 \text{ interval.}$$



$$\begin{aligned} \underline{C_k \times d_i^m} &= \begin{cases} C_k & (k \leq i) \\ C_{k-1} & (k > i) \end{cases} \\ &= C_k \times d_i^m \end{aligned}$$

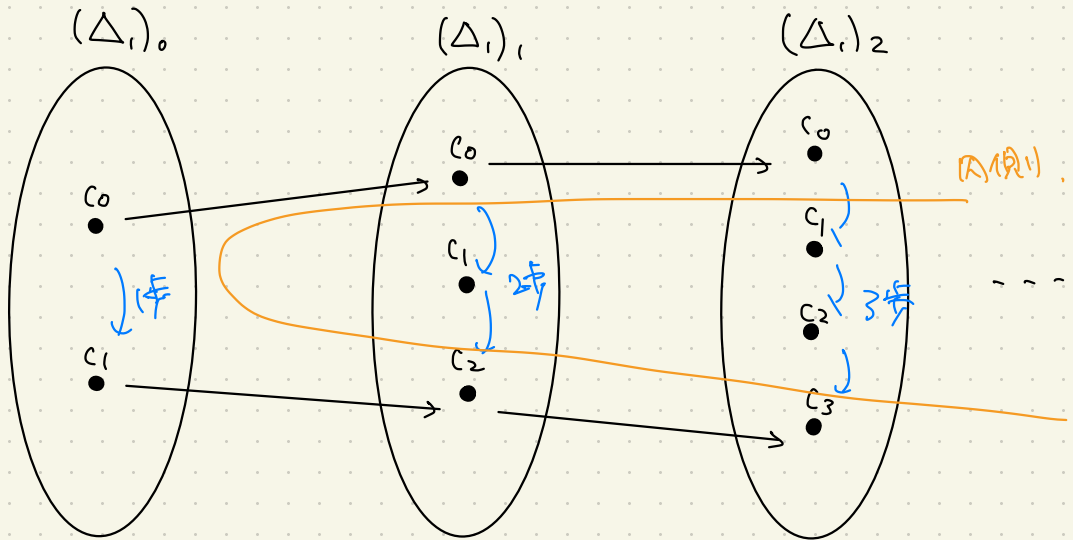
$$\begin{aligned} C_k \times S_i^m &= \begin{cases} C_k & (k \leq i) \\ C_{k+1} & (k > i) \end{cases} \\ &= C_k \times S_i^m \end{aligned}$$



$$\Delta_0 \xrightleftharpoons[i_1]{i_0} \Delta_1$$

$$\Delta_0 \xrightarrow{i_0} \Delta_1 \text{ に対して } * \mapsto c_{n+1}$$

$$\Delta_0 \xrightarrow{i_1} \Delta_1 \text{ に対して } * \mapsto c_0$$



2つの Δ_0 の間に $(\Delta_1)_n$ まで 元から n 個の要素、2つ子。