TOPOS-THEORETIC APPROACH TO REPRESENTATION THEORY, FOR NON-LOGICIANS

RYUYA HORA

ABSTRACT. This is a note on my topos-theoretic approach to representation theory, intended for mathematicians with no background in logic.

CONTENTS

In this note, k denotes a (commutative) field.

1. PRELIMINARIES ON TOPOS THEORY

1.1. topos.

Definition 1.1. A category $\mathbf{E}$ is a (Grothendieck) **topos** if there exists a small site $(\mathcal{C}, J)$ such that $\mathbf{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$.

Example 1.2 (Presheaves)**.** Every presheaf category $\mathbf{PSh}(\mathcal{C})$ over a small category $\mathcal{C}$ is a topos. Examples include

- $\mathbf{PSh}(G)$ for a group.
- $\mathbf{PSh}(FQ)$ for a free category of a quiver Q.
- $\mathbf{PSh}(\mathcal{M}at_k)$ for the category of matrices with coefficients in k (whose objects are non-negative integers)[1].

Example 1.3 (Sheaves on a space)**.** For a topological space X (or, more naturally, a locale X), its sheaf category $\mathbf{Sh}(X)$ is a topos.

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Graduate School of Mathematical Sciences, University of Tokyo. hora@ms.u-tokyo.
[1]In terms of categorical logic, this topos is what's called **the classifying topos of the theory of k-vector spaces.**

Example 1.4 (Topological group)**.** For a topological group (G, τ), the full subcategory of $\mathbf{PSh}(G)$ consisting of all continuous actions $X \times G \to X$ (where a set X is regarded as a discrete topological space) is a topos, which will be denoted by $\mathbf{Cont}(G, \tau)$. In other words, an action $X \times G \to X$ belongs to $\mathbf{Cont}(G, \tau)$ if and only if the stabilizer subgroup is open for every element $x \in X$.

1.2. **Geometric morphism.**

Definition 1.5. A **geometric morphism** from a topos $\mathbf{E}$ to another topos $\mathbf{F}$ is an adjunction $f^* \dashv f_*$ with

- a functor $f_* \colon \mathbf{E} \to \mathbf{F}$, and
- a finite limit preserving functor $f^* \colon \mathbf{F} \to \mathbf{E}$.

A **transformation** from a geometric morphimsms $f \colon \mathbf{E} \to \mathbf{F}$ to $g \colon \mathbf{E} \to \mathbf{F}$ is a natural transformation $f^* \Rightarrow g^*$.

Remark 1.6. These data, namely topoi, geometric morphisms, and transformations, form a 2-category of topoi $\mathbf{Topoi}$. This 2-categorical nature allows us to consider the Hom-category (not Hom-set), which will play a central role in the following content.

For two topoi $\mathbf{E}, \mathbf{F}$, the (1-)category of geometric morphisms $\mathbf{E} \to \mathbf{F}$ and transformations is denoted by $\mathbf{Topoi}(\mathbf{E}, \mathbf{F})$.

Example 1.7. A continuous map from a topological space X to another topological space Y induces a geometric morphism $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ (with the usual direct/inverse image adjunction). This construction provides an almost fully faithful embedding of topology into topos theory[2].

2. Topos-theoretic presentation of representation

Just to make it simple, we adopt the follwong notation:

Notation 2.1. $\mathbf{X}_k \coloneqq \mathbf{PSh}(\mathcal{M}at_k)$

Geometrically, the topos $\mathbf{X}_k$ is **the moduli space of k-vector spaces**. Its points are vector spaces

$$\mathbf{Topoi}(\mathbf{Set} \simeq \mathbf{Sh}(1), \mathbf{X}_k) \simeq \mathbf{Vect}_k,$$

and its path are linear functions

$$\mathbf{Topoi}(\mathbf{PSh}(\to), \mathbf{X}_k) \simeq \mathbf{Vect}_k^{\to}.$$

The categorical-logical universality of the topos $\mathbf{X}_k$, which is usually stated as

$$\mathbf{Topoi}(\mathbf{E}, \mathbf{X}_k) \simeq \mathbf{Vect}_k(\mathbf{E}),$$

immediately implies that the following topos-theoretic presentation of several types of representation theory:

Proposition 2.2. *We have the following equivalence of categories:*

- *For a group G,*
$$\mathbf{Topoi}(\mathbf{PSh}(G), \mathbf{X}_k) \simeq \mathbf{Rep}_k(G).$$
- *For a quiver Q,*
$$\mathbf{Topoi}(\mathbf{PSh}(FQ), \mathbf{X}_k) \simeq \mathbf{Rep}_k(Q).$$
- *For a topological group (G, τ),*
$$\mathbf{Topoi}(\mathbf{Cont}(G, \tau), \mathbf{X}_k) \simeq \mathbf{Rep}_k^{smooth}(G, \tau).$$
- *For a topological space X,*
$$\mathbf{Topoi}(\mathbf{Sh}(X), \mathbf{X}_k) \simeq \text{the category of } \mathbf{Vect}_k\text{-valued sheaves over } X.$$

[2]This statement is verified by the 2-category of locales.

Proposition 2.2 realizes a representaition as a “continuous map” from a represented structure (like a group, or a quiver) to the moduli space of vector spaces.

Remark 2.3 (Digression: Classifying topos)**.** One of the appeals of topos-theoretic generalization of the notion of spaces is that we can construct such a big moduli space (of models of a given theory), called **the classifying topos of a theory**. This construction works not only fot the theory of vector spaces, but also for “almost all theory,” and this is a path to the categorical model theory ([MM94]).

Theory	$\leftrightarrow$	Moduli space
true	$\leftrightarrow$	**Set**
false	$\leftrightarrow$	$\mathbf{1}$
$\forall x, y,\ x = y$	$\leftrightarrow$	$\mathbf{PSh}(\to)$
k-vect. sp.	$\leftrightarrow$	$\mathbf{X}_k$
local ring	$\leftrightarrow$	**Zariski**
interval	$\leftrightarrow$	**sSet**

Table 1. Examples of classifying topoi

memo: Bernstein’s idea [Ber14] might be related.
memo: Dold-Kan correspondence provides another interesting example:

$$\mathbf{Topoi}(\mathbf{sSet}, \mathbf{X}_k) \simeq \mathbf{Ch}^+(\mathbf{Vect}_k).$$

3. Why is it interesting? Towards a conceptual proof of Gabriel’s theorem

I will explain why the topos-theoretic approach to representation theory is appealing (to the author).

3.1. **Topos as a universe.** This approach originally stemmed from a motivation to rewrite Gabriel’s theorem in the quiver representation theory. When the author attended a course on cluster algebras and quiver representations, the author was astonished to find how **Gabriel’s theorem appeared almost in the form of categorical logic**. To explain this, we must first touch upon the logical aspect of topos theory.

The charm of topos lies in its multifaceted nature. Simply put, a topos is both a space and a mathematical universe. Grothendieck originally defined a topos as a space[3]. Later, Lawvere and Tierney discovered that a topos is a category with enough structure to develop mathematics within it. For example, one can define natural numbers, construct integers and rational numbers, build real numbers through Dedekind cuts, and ultimately arrive at complex numbers—all within a topos. The complex numbers constructed in this way exhibit behaviors similar to, yet distinct from, the usual complex numbers in different topoi (such as the case when constructing Dedekind cuts in the topos of sheaves on a topological space, where a ring of complex-valued continuous functions appears). Informally speaking, each topos is a consistent yet uniquely individual parallel world, and this astonishing fact is unfortunately almost unknown, likely due to the lack of ‘geometers $\wedge$ logicians’ capable of handling it. The idea of “topoi as parallel worlds,” however, has been implicitly or explicitly used in various contexts. Cohen’s forcing method for proving the independence of the Continuum Hypothesis later turned out to be fully expressible as a construction of what is now called Cohen topos. Recent advances in Galois theory of differential schemes and condensed mathematics are also examples.

[3]Grothendieck said: *il faut le considérer comme étant plus ou moins l’équivalent du terme “espace”* in his ‘Récoltes et Semailles.’

> [Tom20, Changing the universe] An established principle in topos theory states that the universe of sets can be replaced by an arbitrary base topos, and that it should be possible to reprise interesting topics and chapters of classical mathematics in the new context.
>
> We cling to this principle as a kind of Ariadne's thread, guiding us out of the labyrinth of guesswork on the path of applying the vast machinery of topos theory and categorical logic in the case of difference sets. There is no need to wonder how to define appropriate difference analogues of classical objects, a predicament often encountered by a researcher in difference algebra.

3.2. **Different representation theories = The single linear algebra in different universes.** A logical interpretation of proposition 2.2 is that a representation is a model of the theory of k-vector spaces in a topos. Informally, **a representation is just a vector space in a parallel world[4]!**

Of cource, different worlds have different logical characteristics. For example, the group action topos $\mathbf{PSh}(G)$ satisfies both the axiom of choice and the law of excluded middle, while the topological group action topos $\mathbf{Cont}(G, \tau)$ satisfies the law of excluded middle but not the axiom of choice. The quiver action topos $\mathbf{PSh}(FQ)$ does not even satisfy the law of excluded middle. These logical characteristics of a topos naturally influence its internal mathematics, in particular, internal linear algebra (see [Ble21]). The inhabitants of different universes, even if they develop linear algebra in the same way, sometimes arrive at different results depending on the logical structure of their universes. **Such subtleties of logical structures in parallel worlds then manifest to us as various types of representation-theoretic properties in the ordinary mathematics!**

My insight into Gabriel's theorem is that it is explained by the logical characteristics of the quiver action topoi. In other words, different quivers generate different topoi, and what Gabriel's theorem asserts is that "the necessary and sufficient condition for there being finitely many finite-dimensional irreducible vector spaces within the quiver action topos is that the quiver is a Dynkin diagram." To reprase this, let us call a topos in which the internal mathematics satisfies "there are finitely many finite-dimensional indecoposable vector spaces" **a Dynkin topos** (or, a Dynkin world, if you prefer). For example, our usual universe **Set** is a Dynkin topos, since the only finite-dimensional irreducible vector space is the one-dimentional vector space k. Gabriel's theorem characterizes Dynkin diagrams as the necessary and sufficient condition for a quiver action topos to being a Dynkin topos [5].

3.3. **Conclusion.** In the end, our idea is that "subtle differences between different representation theories reflect the logic of the corresponding parallel worlds (via the behavior of internal linear algebra)." This is just one example of a broader idea of topos theory: "Different topoi are parallel worlds with different individualities, and 'theories with subtle differences' are often a single theory interpreted in different topoi." Knowing this, I find it hard to believe that the fact that the representation theory of finite groups has better properties than that of quivers is unrelated to the fact that finite group action topoi satisfy more logical axioms than quiver action topoi.

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[4] Form now on, the words 'universe,' 'world,' 'parallel world,' and 'space,' all refer to a topos.

[5] If a simple axiom A, necessary and sufficient to prove "there are finitely many finite-dimensional irreducible vector spaces," is found via (constructive) reverse mathematics, the logical characterization of Dynkin diagrams will become even more apparent.