

TOPOI OF AUTOMATA III: GEOMETRY OF Σ -Set

RYUYA HORA

ABSTRACT. The aim of this paper is to provide a concrete description of the category of points of Σ -Set (theorem 1.12).

This work is in progress and does not constitute the complete version of the project.

CONTENTS

1. An example of topos-geometry: Points of Σ -Set	1
1.1. Points as trees	2
References	4

1. AN EXAMPLE OF TOPOS-GEOMETRY: POINTS OF Σ -Set

In this paper, we will fix an alphabet Σ . Let Σ^* denote the set of all words, i.e., the free monoid over the set Σ .

Definition 1.1. We adopt the following definitions and terminologies.

- A Σ -set is a (possibly infinite) set Q equipped with a function $\delta: Q \times \Sigma \rightarrow Q$.
- An element of Q is called a **state**, and the associated function δ is called the **transition function**.
- A morphism of Σ -sets $f: (Q_1, \delta_1) \rightarrow (Q_2, \delta_2)$ is a function $f: Q_1 \rightarrow Q_2$ that commutes with their transition functions.

$$\begin{array}{ccc} Q_1 \times \Sigma & \xrightarrow{\delta_1} & Q_1 \\ \downarrow f \times \text{id}_\Sigma & & \downarrow f \\ Q_2 \times \Sigma & \xrightarrow{\delta_2} & Q_2 \end{array}$$

- The category of Σ -sets is denoted by Σ -Set.

The following (seemingly boring) proposition is our starting point:

Proposition 1.2. *The category of Σ -sets is equivalent to category of right Σ^* -actions*

$$\Sigma\text{-Set} \simeq \mathbf{PSh}(\Sigma^*).$$

In particular, it is a presheaf topos (and hence, a Grothendieck topos).

Date: September 20, 2026.

2020 Mathematics Subject Classification. 18F10, 68Q70, 20M35, 18B20.

Key words and phrases. Automaton, topos, regular language, coalgebra, finite monoid, profinite word, Myhill-Nerode theorem.

ZEN University. ryuya_hora@zen.ac.jp.

Proof. For a Σ -set (Q, δ) , the right action of a word $w = a_1 \dots a_n$ on $q \in Q$ is defined by

$$qw := \delta(\dots \delta(\delta(q, a_1), a_2), \dots a_n).$$

This construction naturally induces an equivalence of categories $\Sigma\text{-Set} \rightarrow \mathbf{PSh}(\Sigma^*)$. $\square$

This terminology is due to [Rog19].

Definition 1.3. *The canonical point* $p: \mathbf{Set} \rightarrow \Sigma\text{-Set}$ of the topos $\Sigma\text{-Set}$ is the geometric morphism

$$p: \mathbf{Set} \simeq \mathbf{PSh}(1) \rightarrow \mathbf{PSh}(\Sigma^*) \simeq \Sigma\text{-Set},$$

induced by the unique monoid homomorphism $1 \rightarrow \Sigma^*$, where the inverse image functor p^* is the forgetful functor.

Definition 1.4. A left Σ^* -set $(V, \epsilon: \Sigma \times V \rightarrow V)$ is **flat**, if it satisfies the following equivalent conditions:

- The tensor product functor $\mathbf{Set} \xrightarrow{\bar{\otimes}_{\Sigma^*} V} \Sigma\text{-Set}$ is left exact.
- It satisfies the following conditions:
 - (0): V is non-empty.
 - (1): For any two elements $v_1, v_2 \in V$, there exists an element $v \in V$ and two words $w_1, w_2 \in \Sigma^*$ such that $w_1 v = v_1$ and $w_2 v = v_2$.
 - (2): For any element $v \in V$ and two words w_1, w_2 such that $w_1 v = w_2 v$, we have $w_1 = w_2$.

The equivalence between the two definitions above is a special case of a well-known fact that a functor $\mathcal{C} \rightarrow \mathbf{Set}$ is filtering if and only if flat. See [MM94, Chapter VII, Section 6] for detailed proof.

Proposition 1.5. *The category of points of $\Sigma\text{-Set}$ is equivalent to the category of flat left Σ^* -sets.*

Proof. This is a special case of the Diaconescu theorem. $\square$

Example 1.6 (The canonical point). *The free left Σ -set* $(\Sigma^*, \Sigma \times \Sigma^* \rightarrow \Sigma^*: (a, w) \mapsto aw)$ is flat. It corresponds to the canonical point of $\Sigma\text{-Set}$ (Definition 1.3).

The first observation is that ϵ is bijective in most cases:

Lemma 1.7. *If a flat left Σ -set $(V, \epsilon: \Sigma \times V \rightarrow V)$ is not isomorphic to the free left Σ -set (Example 1.6), the action map $\epsilon: \Sigma \times V \rightarrow V$ is bijective.*

Proof. First, we prove that ϵ is injective. Suppose $av_1 = bv_2$ for $a, b \in \Sigma$ and $v_1, v_2 \in V$. We will prove $a = b$ and $v_1 = v_2$. Since V is flat, we can take w_1, w_2 and $v \in V$ such that $w_1 v = v_1$ and $w_2 v = v_2$. Then we have $aw_1 v = bw_2 v$, and the flatness implies $aw_1 = bw_2$. This proves $a = b$, $w_1 = w_2$, and $v_1 = v_2$.

Suppose ϵ is not surjective, and take an element $v \in V$ such that $v \notin \text{Im}(\epsilon)$. The flatness implies that, for every element $v' \in V$, there exists $v'' \in V$ and $w, w' \in \Sigma^*$ such that $v' = w'v''$ and $v = wv''$. However, since v is not in the image of ϵ , w should be the empty word, and we obtain $v' = w'v$. This implies that $\Sigma^* \rightarrow V: w \mapsto wv$ is surjective. The flatness condition implies that $\Sigma^* \rightarrow V$ is actually a bijection, and it proves that V is isomorphic to the free left Σ -set. $\square$

1.1. Points as trees. Our remaining task is to determine when a bijection $\epsilon: \Sigma \times V \rightarrow V$ provides a flat left Σ -set. Our approach to this task is utilizing (infinite) graphs!

Definition 1.8. We adopt the following terminologies:

- A Σ -labeled **graph** is a tuple $(V, E, s: E \rightarrow V, t: E \rightarrow V, l: E \rightarrow \Sigma)$.
- A Σ -labeled **graph morphism** is a graph morphism that preserves the labels.
- For a set $S \subset \mathbb{N} \cup \{\infty\}$, a Σ -labeled graph is said to be (S, Σ) -**graph**, if

- for each vertex $v \in V$ and $a \in \Sigma$, there exists a unique a -labeled edge e out of v (i.e. $s(e) = v$ and $l(e) = a$).
- for each vertex $v \in V$, its indegree belongs to S .

$$\#\{e \in E \mid t(e) = v\} \in S$$

Definition 1.9 (Associated graph). For a left Σ -set $(V, \epsilon: \Sigma \times V \rightarrow V)$, we define the **associated graph** by

Vertex: draw a vertex for each element of V

Edge: for each element $v \in V$ and $a \in \Sigma$, we draw an edge from v to av , with a label a .

This construction defines a functor from the category of left Σ -sets to the category of Σ -labeled graphs.

Lemma 1.10. *We obtain the following equivalences of categories:*

- The category of $(\mathbb{N} \cup \{\infty\}, \Sigma)$ -graphs is equivalent to the category of left Σ -sets¹.
- The category of $(\{0, 1\}, \Sigma)$ -graphs is equivalent to the full subcategory of left Σ -sets, consisting of left Σ -set $(X, \epsilon: \Sigma \times X \rightarrow X)$ with injective ϵ .
- The category of $(\{1\}, \Sigma)$ -graphs is equivalent to the full subcategory of left Σ -sets, consisting of left Σ -set $(X, \epsilon: \Sigma \times X \rightarrow X)$ with bijective ϵ .

Proof. This immediately follows from the definition. $\square$

Definition 1.11. A Σ -labeled graph is a **tree**, if its underlying undirected graph is connected (in particular non-empty) and does not contain cycles. If it is also a (S, Σ) -graph, we call it a (S, Σ) -**tree**.

Theorem 1.12. *We obtain the following equivalences of categories:*

- The category of points of Σ -**Set** is equivalent to the category of $(\{0, 1\}, \Sigma)$ -trees.
- The category of non-canonical points of Σ -**Set** is equivalent to the category of $(\{1\}, \Sigma)$ -trees.

To prove this theorem, it is enough to prove the following lemma.

Lemma 1.13. *For a left Σ -set $(V, \epsilon: \Sigma \times V \rightarrow V)$ with injective ϵ , it is flat if and only if the corresponding $(\{0, 1\}, \Sigma)$ -graph is a $(\{0, 1\}, \Sigma)$ -tree.*

Proof. First, we prove the ‘only-if’ part. The flatness implies that the corresponding graph is (non-empty and) connected. So, what we should show is that the corresponding $(\{0, 1\}, \Sigma)$ -graph does not contain cycles. Suppose that there is a cycle C in the underlying undirected graph. Then, the flatness implies that all (finitely many) vertices in the cycle C are in the orbit of an element $v \in V$. However, the flatness also implies that the orbit of v is a tree, which contradicts the existence of the cycle C .

Secondly, we prove the ‘if’ part. Suppose the corresponding graph is a tree. We will prove the Σ -set is flat. Since the graph is a tree, V is non-empty. For two elements $v_1, v_2 \in V$, we can take a path from v_1 to v_2 , ignoring the direction of edges. Since the indegree of each vertex is at most 1, the zigzag path should be of the form of $v_1 \leftarrow \cdots \leftarrow \rightarrow \cdots \rightarrow v_2$ (not containing $\rightarrow \leftarrow$). This proves that there is an element v and $w_1, w_2 \in \Sigma^*$ such that $w_1 v = v_1$ and $w_2 v = v_2$. So far, we have proven the first two conditions of flatness. The last condition immediately follows from the assumption that the graph is a tree. $\square$

Notice that

- Lemma 1.7 implies that the $(\{0, 1\}, \Sigma)$ -tree that is not $(\{1\}, \Sigma)$ -tree is unique up to isomorphism.
- All $(\{1\}, \Sigma)$ -trees are mutually isomorphic as unlabeled directed graphs. The only differences are labels.

Example 1.14 (“Diachrony and Synchrony”). Consider the case where Σ is a singleton. In this case, the topos Σ -**Set** is *the topos of discrete dynamical systems*. Theorem 1.12 states that this topos has exactly 2 points:

¹It is also equivalent to the category of right Σ -sets

Canonical: One is the canonical point, corresponding to the free left Σ -set $(\mathbb{N}, +1)$, and to the $(\{0, 1\}, \Sigma)$ -tree

$$\cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdots .$$

Non-canonical: The other is the unique non-canonical point, corresponding to the flat left Σ -set $(\mathbb{Z}, +1)$, and to the $(\{1\}, \Sigma)$ -tree

$$\cdots \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdots .$$

REFERENCES

- [MM94] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 1994.
- [Rog19] Morgan Rogers. “Toposes of Discrete Monoid Actions”. *arXiv:1905.10277* (2019).