

TOPOS WITH ENOUGH PROJECTIVES

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ABSTRACT. In this paper, we provide a site characterization of topoi with enough projectives. More concretely, we prove that a Grothendieck topos has enough projective objects if and only if it is a sheaf topos over a κ -extensive site for some regular cardinal κ . We will see that 2-Gaeta topoi are precisely presheaf topoi.

- The question is asked at [\[mathoverflow 316588\]](#) by Morgan Rogers.

This work is in progress and does not constitute the complete version of the project.

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1. κ -EXTENSIVE SITE AND κ -GAETA TOPOI

For a (possibly finite) cardinal κ , we mean ‘less than κ ’ by the word ‘ κ -small.’

Definition 1.1. A (possibly finite) cardinal κ is **regular**, if

- $1 \in \kappa$, and
- for any κ -small family of κ -small sets $\{X_i\}_{i \in I}$,

$$|I| < \kappa \text{ and } \forall i \in I \ |X_i| < \kappa$$

their sum $\coprod_{i \in I} X_i$ is also κ -small.

The sequence of regular cardinals starts with $\kappa = 2, \aleph_0, \aleph_1, \dots, \aleph_{\omega+1}, \dots$

Definition 1.2. For a regular cardinal κ , a small category $\mathcal{C}$ is said to be κ -extensive, if $\mathcal{C}$ admits

- κ -small coproducts, and
- pullbacks of κ -small coproduct inclusion maps (along arbitrary maps),

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and furthermore, κ -small coproducts are

- disjoint, and
- pullback stable.

$\kappa = 2$: A category $\mathcal{C}$ is 2-extensive, if and only if $\mathcal{C}$ has a strict initial object.

$\kappa = \aleph_0$: The $\aleph_0$ -extensivity is the usual extensivity as studied in [CLW93].

Notation 1.3. For a regular cardinal κ and a small κ -extensive category $\mathcal{C}$, we write $J_{\kappa\text{-ext}}$ for the collection of sieves S that contain a κ -small coproduct diagram. In other words, a sieve S belongs to $J_{\kappa\text{-ext}}$ if and only if there is a κ -small family $\{X_i \rightarrow X\}_{i \in I} \subset S$ such that the canonical map

$$\coprod_{i \in I} X_i \rightarrow X$$

is an isomorphism.

Lemma 1.4 (Description of κ -extensive topology [Shu12, Example 11.12]). *For a regular cardinal κ and a small κ -extensive category $\mathcal{C}$, the collection $J_{\kappa\text{-ext}}$ is a Grothendieck topology on $\mathcal{C}$.*

Proof. We need to check the three axioms.

Trivial cover: For any object $X \in \text{ob}(\mathcal{C})$, the trivial sieve $\langle \text{id}_X \rangle$ belongs to $J_{\kappa\text{-ext}}$, since $\text{id}_X: X \rightarrow X$ is a κ -small coproduct diagram. (We used the assumption $1 \in \kappa$.)

Stability: This follows from the κ -extensivity of $\mathcal{C}$, especially from the condition that κ -small coproducts are pullback stable.

Transitivity: This follows from (the second condition of) the regularity of κ .

□

Definition 1.5. For a regular cardinal κ and a small κ -extensive category $\mathcal{C}$, we call the Grothendieck topology $J_{\kappa\text{-ext}}$ (Notation 1.3, Lemma 1.4) **the κ -extensive topology** associated to $\mathcal{C}$. We call the site $(\mathcal{C}, J_{\kappa\text{-ext}})$ **the κ -extensive site** associated to $\mathcal{C}$.

Definition 1.6 (κ -Gaeta topos). For a regular cardinal κ , we call a Grothendieck topos $\mathcal{E}$ a **κ -Gaeta topos**, if there exists a small κ -extensive category $\mathcal{C}$ such that $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J_{\kappa\text{-ext}})$.

Proposition 1.7. *For a regular cardinal κ and a small κ -extensive category $\mathcal{C}$, a presheaf $T: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ is a $J_{\kappa\text{-ext}}$ -sheaf if and only if it sends κ -small coproducts to κ -small products.*

$$T\left(\coprod_{i \in I} X_i\right) \cong \prod_{i \in I} T(X_i)$$

Proof. We first assume that T is a $J_{\kappa\text{-ext}}$ -sheaf, and prove that T preserves κ -small coproducts. Since the empty sieve covers the strict initial object $\emptyset \in \text{ob}(\mathcal{C})$, we have $T(\emptyset) = 1$. For a κ -small coproduct diagram $\{X_i \rightarrow X\}_{i \in I}$, let S denote the sieve on X generated by the coproduct inclusions. By the κ -extensivity of $\mathcal{C}$ (especially by the

disjointness of κ -small coproducts), the sieve $S \mapsto \mathcal{J}(X)$ (regarded as an object of $\mathbf{PSh}(\mathcal{C})$) is the pushout of

$$\begin{array}{c}
 & & \mathcal{J}(X_i) \\
 & \nearrow & \\
 \mathcal{J}(\emptyset) & \xrightarrow{\quad} & \mathcal{J}(X_{i'}) \\
 & \searrow & \\
 & & \mathcal{J}(X_{i''}) \\
 & & \vdots
 \end{array}$$

in the presheaf topos $\mathbf{PSh}(\mathcal{C})$. Thus the glueing condition of $S \mapsto \mathcal{J}(X)$ states that

$$\begin{array}{ccccc}
 & & T(X_i) & & \\
 & \nearrow & & \searrow & \\
 T(X) & \longrightarrow & T(X_{i'}) & \longrightarrow & T(\emptyset) \\
 & \searrow & & \nearrow & \\
 & & T(X_{i''}) & & \\
 & & \vdots & &
 \end{array}$$

is a pullback diagram (in $\mathbf{Set}$). Thus, $T(\emptyset) \cong 1$ implies $T(X) \cong \prod_{i \in I} T(X_i)$.

Conversely, we assume that T sends κ -small coproducts to κ -small products. We prove that T is a $J_{\kappa\text{-ext}}$ -sheaf. Let S be a $J_{\kappa\text{-ext}}$ -covering sieve on an object $X \in \text{ob}(\mathcal{C})$, and let $\{t_f \in T(Y)\}_{f: Y \rightarrow X \in S}$ be an S -matching family. By definition of $J_{\kappa\text{-ext}}$, it contains a κ -small coproduct diagram $\{\iota_i: X_i \rightarrow X\}_{i \in I}$. By the assumption, we obtain $t \in T(X)$ that corresponds to (t_{ι_i}) via the canonical bijection $T(X) \cong \prod_{i \in I} T(X_i)$. In other words, $t \in T(X)$ is the unique element such that $t_{\iota_i} = t_{\iota_i}$ for every $i \in I$.

Since t is the unique candidate for the amalgamation of the $\{t_f\}_{f \in S}$, it suffices to prove $t_f = t_{f \circ \tau_i} \in T(Y)$ for any $f: Y \rightarrow X \in S$. We define Y_i by the following pullback square

$$\begin{array}{ccc}
 Y_i & \xrightarrow{f_i} & X_i \\
 \downarrow \tau_i & \lrcorner & \downarrow \iota_i \\
 Y & \xrightarrow{f} & X
 \end{array}$$

(Notice that the existence of pullbacks of that shape is ensured by the κ -extensivity.) Since we have $T(Y) \cong \prod_{i \in I} T(Y_i)$, it suffices to prove that $(t_f)\tau_i = (t_{f \circ \tau_i})\tau_i$ for every $i \in I$. This holds since $(t_f)\tau_i = t_{f \circ \tau_i} = t_{\iota_i f_i} = t_{\iota_i} f_i = t_{\iota_i} f_i = (t_{f \circ \tau_i})\tau_i$. $\square$

Corollary 1.8. *For a regular cardinal κ and a small κ -extensive category $\mathcal{C}$, the κ -extensive topology $J_{\kappa\text{-ext}}$ is subcanonical.*

Proof. This immediately follows from Proposition 1.7 □

The case where $\kappa = 2$ is worth mentioning:

Corollary 1.9. *A Grothendieck topos $\mathcal{E}$ is a 2-Gaeta topos if and only if it is a presheaf topos.*

Proof. For any small category $\mathcal{C}$, we obtain the category $\mathcal{C}^\triangleleft$ by formally adding a new initial object to $\mathcal{C}$. Since the 2-extensivity is equivalent to the existence of a strict initial object, the resulting category $\mathcal{C}^\triangleleft$ is 2-extensive. Conversely, one can easily observe that every 2-extensive category $\mathcal{D}$ is of the form of $\mathcal{D} \simeq \mathcal{C}^\triangleleft$ with a small category $\mathcal{C}$. Then, the equivalence of categories

$$\mathbf{Sh}(\mathcal{C}^\triangleleft, J_{2\text{-ext}}) \simeq \mathbf{PSh}(\mathcal{C})$$

provided by Proposition 1.7 completes the proof. □

2. FROM κ -EXTENSIVE SITE TO PROJECTIVE OBJECTS

2.1. Sites generated by free sieves.

Definition 2.1. For a small category $\mathcal{C}$, we say a sieve S on an object X is **free** if the sieve S , regarded as a presheaf on $\mathcal{C}$, is a small coproduct of representable presheaves.

In other words, a sieve S is free if and only if there is a family of elements $\{f_i: X_i \rightarrow X\}_{i \in I} \subset S$ satisfying the following equivalent conditions:

- (1) Every element $f: Y \rightarrow X \in S$ is decomposed into

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow \exists! h & \nearrow \exists! i \in I f_i \\ & X_i & \end{array}$$

with a unique pair of $i \in I$ and $h: Y \rightarrow X_i$.

- (2) In the presheaf category $\mathbf{PSh}(\mathcal{C})$, the canonical morphism

$$\coprod_{i \in I} \mathcal{Y}(X_i) \rightarrow S$$

is an isomorphism.

- (3) For any presheaf F , the morphism

$$\mathbf{PSh}(\mathcal{C})(S, F) \rightarrow \prod_{i \in I} F(X_i)$$

is bijective.

Remark 2.2. If $\mathcal{C}$ is Cauchy complete, a sieve (or more generally, any presheaf) S is free if and only if S is projective.

Definition 2.3. For a small category $\mathcal{C}$, we say a Grothendieck topology J is **generated by free sieves** if, for any J -covering sieve S , there exists a free J -covering subsieve $S' \subset S$.

Example 2.4. The trivial topology $(\mathcal{C}, J_{\text{triv}})$ is generated by free sieves. In fact, the maximum sieve on each object X is free.

Proposition 2.5. *For a regular cardinal κ and a κ -extensive category $\mathcal{C}$, we consider its full subcategory $\mathcal{C}'$ that has all objects in $\mathcal{C}$ except the initial object. Then the restriction of $J_{\kappa\text{-ext}}$ on $\mathcal{C}'$, which is denoted by $J'_{\kappa\text{-ext}}$, is a Grothendieck topology on $\mathcal{C}'$ generated by free sieves. Furthermore, we have $\mathbf{Sh}(\mathcal{C}, J_{\kappa\text{-ext}}) \simeq \mathbf{Sh}(\mathcal{C}', J'_{\kappa\text{-ext}})$.*

Proof. Since $(\mathcal{C}, J_{\kappa\text{-ext}})$ is subcanonical (Corollary 1.8) and the initial object is covered by the empty sieve, the comparison lemma [MM92, Appendix, Section 4, Corollary 3] implies that $(\mathcal{C}', J'_{\kappa\text{-ext}})$ is a site and $\mathbf{Sh}(\mathcal{C}, J_{\kappa\text{-ext}}) \simeq \mathbf{Sh}(\mathcal{C}', J'_{\kappa\text{-ext}})$. We prove that the site $(\mathcal{C}', J'_{\kappa\text{-ext}})$ is generated by free sieves. If a sieve S on an object $X \in \text{ob}(\mathcal{C}')$ is a $J'_{\kappa\text{-ext}}$ -covering sieve, then S contains a κ -small coproduct diagram $\{\iota_i: X_i \rightarrow X\}_{i \in I} \subset S$. Let $S' \subset S$ be the subsieve generated by the family $\{\iota_i: X_i \rightarrow X\}_{i \in I}$. For any morphism $f: Y \rightarrow X \in S'$, f factors through a unique ι_i , since Y is not initial in $\mathcal{C}$ and κ -small coproducts are disjoint in $\mathcal{C}$. This proves that S' is a free subsieve of S . $\square$

2.2. Site to projective objects.

Definition 2.6. We say that a site $(\mathcal{C}, J)$ **witnesses** that a Grothendieck topos $\mathcal{E}$ has enough projectives if we have $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ and the inclusion functor

$$\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow \mathbf{PSh}(\mathcal{C})$$

preserves epimorphisms.

Notice that a site $(\mathcal{C}, J)$ witnesses its sheaf topos $\mathbf{Sh}(\mathcal{C}, J)$ has enough projectives if and only if, for each $x \in \text{ob}(\mathcal{C})$, its represented sheaf $\mathbf{a}\mathcal{K}(x)$ is projective. A Grothendieck topos $\mathcal{E}$ has enough projective objects, if and only if it admits a site $(\mathcal{C}, J)$ that witnesses that $\mathcal{E}$ has enough projectives.

Lemma 2.7 (Epimorphisms in a sheaf topos). *For a site $(\mathcal{C}, J)$, a morphism of sheaves $f: A \rightarrow B$ is epic in the topos $\mathbf{Sh}(\mathcal{C}, J)$ if and only if, for any $X \in \text{ob}(\mathcal{C})$ and $b \in B(X)$, there exists a J -covering sieve S and a (not necessarily S -matching) family $\{a_h \in A(Y)\}_{h: Y \rightarrow X \in S}$, such that $bh = f_Y(a_h) \in B(Y)$ for any $h: Y \rightarrow X \in S$.*

Proof. The morphism f is an epimorphism if and only if its image coincides with B . Since the image of f in the sheaf topos is the sheafification of the objectwise image, we obtain the above description. $\square$

Proposition 2.8. *If a site $(\mathcal{C}, J)$ is generated by free sieves, then $(\mathcal{C}, J)$ witnesses that $\mathbf{Sh}(\mathcal{C}, J)$ has enough projectives.*

Proof. Suppose that the site $(\mathcal{C}, J)$ is generated by free sieves. Let $f: A \rightarrow B$ be an epimorphism in the topos $\mathbf{Sh}(\mathcal{C}, J)$. Take an arbitrary object $X \in \text{ob}(\mathcal{C})$ and $b \in B(X)$. We will construct $a \in A(X)$ such that $f_X(a) = b$.

Since f is epic, Lemma 2.7 provides a J -covering sieve S on the object X , and $\{a_h \in A(Y)\}_{h: Y \rightarrow X \in S}$, such that $bh = f_Y(a_h) \in B(Y)$ for any $h: Y \rightarrow X$. Using the assumption that $(\mathcal{C}, J)$ is generated by free sieves, we can take a free J -covering subsieve $S' \subset S$, and a family of elements $\{h_i: X_i \rightarrow X\}_{i \in I} \subset S'$ that witnesses the isomorphism

$$\coprod_{i \in I} \mathcal{K}(X_i) \cong S'.$$

Since A is a J -sheaf and S' is free, we obtain a bijection

$$A(X) \cong \mathbf{PSh}(\mathcal{C})(\mathcal{K}(X), A) \cong \mathbf{PSh}(\mathcal{C})(S', A) \cong \prod_{i \in I} A(X_i)$$

and the unique element $a \in A(X)$ such that $ah_i = a_{h_i}$ for any $i \in I$.

We prove that $f_X(a) = b$. Since S' is a J -covering, it suffices to prove that $f_X(a)h_i = bh_i$ for each $i \in I$, which is verified by

$$f_X(a)h_i = f_{X_i}(ah_i) = f_{X_i}(a_{h_i}) = bh_i.$$

This completes the proof. $\square$

Corollary 2.9. *If a Grothendieck topos $\mathcal{E}$ is a κ -Gaeta topos for a regular cardinal κ , then $\mathcal{E}$ has enough projectives.*

3. FROM PROJECTIVE OBJECTS TO κ -EXTENSIVE SITES

3.1. Preliminaries on projective objects. We adopt the following terminology.

- A subobject $\iota: S \rightarrowtail X$ is called a **retract** if ι is a split monomorphism.
- A subobject $\iota: S \rightarrowtail X$ is called a **summand** if there exists another subobject $S' \rightarrowtail X$ such that

$$S \rightarrowtail X \leftarrowtail S'$$

is a coproduct diagram.

Lemma 3.1 (Closure properties of projective objects). *For a category $\mathcal{C}$, projective objects satisfy the following closure properties.*

- A retract of a projective object is projective.
- A small coproduct of projective objects is projective.
- If the category $\mathcal{C}$ is extensive, a summand of a projective object is projective.

3.2. A topos with enough projectives has projective generating sets.

Definition 3.2. A category $\mathcal{E}$ has (externally) **enough projectives**, if, for any object $X \in \text{ob}(\mathcal{E})$, there exists a projective object P and an epimorphism $P \twoheadrightarrow X$.

Proposition 3.3 (Morgan Rogers [\[mathoverflow 316588\]](#)). *For a Grothendieck topos $\mathcal{E}$, the following conditions are equivalent:*

- (1) $\mathcal{E}$ has enough projectives.
- (2) $\mathcal{E}$ has a (small) generating set $\mathcal{P}$ consisting of projective objects.

Proof. We first prove (1) $\implies$ (2). Fix a generating set $\mathcal{G}$ of $\mathcal{E}$. If $\mathcal{E}$ has enough projectives for each object $g \in \mathcal{G}$, we can take a projective object P_g equipped with an epimorphism $P_g \twoheadrightarrow g$. This shows that $\mathcal{P} := \{P_g \mid g \in \mathcal{G}\}$ is a generating set consisting of projective objects.

Next, we prove (2) $\implies$ (1). For any object $X \in \mathcal{E}$, the assumption implies that there exists a jointly epimorphic small family of morphisms from projective objects $\{a_\lambda: P_\lambda \rightarrow X\}_{\lambda \in \Lambda}$, where $P_\lambda \in \mathcal{P}$. Lemma 3.1 implies that

$$\coprod_{\lambda \in \Lambda} P_\lambda \twoheadrightarrow X$$

is the morphism from a projective object. □

3.3. κ -narrow objects. To extract a κ -extensive site from a given topos with enough projectives, we consider the notion of being κ -narrow.

Definition 3.4. For a (possibly finite) regular cardinal κ , an object $X \in \text{ob}(\mathcal{E})$ of a Grothendieck topos $\mathcal{E}$ is said to be **κ -narrow** if for any κ -coproduct decomposition

$$X \cong \coprod_{\alpha \in \kappa} X_\alpha$$

there exists $\alpha \in \kappa$ such that X_α is initial.

Example 3.5. An object X is 2-narrow if and only if X is initial or connected.

Example 3.6. For a locally connected topos $\mathcal{E}$, an object X is κ -narrow if and only if $|\pi_0(X)| < \kappa$. In particular, a set X is κ -narrow (in the topos of sets **Set**) if and only if $|X| < \kappa$.

Example 3.7. In the sheaf topos over the Cantor space $\mathbf{Sh}(2^{\mathbb{N}})$, the terminal object is $\aleph_0$ -narrow.

Proposition 3.8 (Closure properties of κ -narrow objects). *For a Grothendieck topos $\mathcal{E}$ and a regular cardinal κ ,*

- A κ -small coproduct of κ -narrow objects is κ -narrow.
- A summand of a κ -narrow object is κ -narrow.

Proof. Let X be a κ -small coproduct of κ -narrow objects $\{X_\lambda\}_{\lambda \in \Lambda}$ ($|\Lambda| < \kappa$).

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

Take an arbitrary κ -coproduct decomposition $X = \coprod_{\alpha \in \kappa} Y_\alpha$. The infinitary extensivity of $\mathcal{E}$ implies that

$$X = \coprod_{\lambda \in \Lambda} X_\lambda \cong \coprod_{\lambda \in \Lambda} \coprod_{\alpha \in \kappa} X_\lambda \times_X Y_\alpha.$$

For each $\lambda \in \Lambda$, we define $I_\lambda := \{\alpha \in \kappa \mid X_\lambda \times_X Y_\alpha \text{ is not initial}\}$. Since each X_λ is κ -narrow, we have $|I_\lambda| < \kappa$. The regularity of κ implies that $\bigcup_{\lambda \in \Lambda} I_\lambda \subsetneq \kappa$. For an element $\alpha \in \kappa \setminus (\bigcup_{\lambda \in \Lambda} I_\lambda)$, we have

$$Y_\alpha \cong \coprod_{\lambda \in \Lambda} X_\lambda \times_X Y_\alpha \cong \coprod_{\lambda \in \Lambda} \emptyset \cong \emptyset.$$

This completes the proof of the former statement.

The latter statement is easier to prove. □

The following lemma is easy, but essential.

Lemma 3.9. *For a regular cardinal κ , a Grothendieck topos $\mathcal{E}$, a κ -narrow projective object P , and a small (not necessarily κ -small) family of morphisms $\{f_i: X_i \rightarrow P\}_{i \in I}$, the following conditions are equivalent:*

- $\{f_i: X_i \rightarrow P\}_{i \in I}$ is jointly epimorphic.
- There exists a κ -small coproduct decomposition $\coprod_{\lambda \in \Lambda} P_\lambda$ ($|\Lambda| < \kappa$) such that every inclusion $P_\lambda \hookrightarrow P$ factors through some $f_i: X_i \rightarrow P$.

Proof. It is easy to prove that the latter condition implies the former. We prove the opposite.

Assume that $\{f_i: X_i \rightarrow P\}_{i \in I}$ is jointly epimorphic. Then we obtain the canonical epimorphism $\sum_{i \in I} f_i: \coprod_{i \in I} X_i \twoheadrightarrow P$. The projectivity of P ensures the existence of a section $\coprod_{i \in I} X_i \leftarrow P: s$. Since $\mathcal{E}$ is infinitary extensive, the morphism s induces the I -coproduct decomposition $P \cong \coprod_{i \in I} P_i$ by the pullback diagram

$$\begin{array}{ccc} X_i & \xleftarrow{\quad} & P_i \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{i \in I} X_i & \xleftarrow{s} & P \\ & \nearrow \sum_{i \in I} f_i & \end{array}$$

This implies that each inclusion $P_i \hookrightarrow P$ factors through $f_i: X_i \rightarrow P$. Since P is κ -narrow, the subset $\Lambda := \{i \in I \mid P_i \not\cong \emptyset\}$ is κ -small, and we have a κ -small coproduct decomposition $P \cong \coprod_{\lambda \in \Lambda} P_\lambda$. This completes the proof. □

3.4. The κ -extensive site of κ -narrow projective objects.

Definition 3.10. For a Grothendieck topos $\mathcal{E}$ and a regular cardinal κ , the full subcategory of κ -narrow projective objects is denoted by $\mathcal{C}_\kappa$.

Example 3.11. For a small category $\mathcal{C}$, the category $\mathcal{C}_2 \subset \mathbf{PSh}(\mathcal{C})$ for $\kappa = 2$ is equivalent to $\overline{\mathcal{C}}^\triangleleft$, where $\overline{\mathcal{C}}$ denotes the Cauchy completion of $\mathcal{C}$. This is the opposite construction of Corollary 1.9.

Theorem 3.12. *For a Grothendieck topos $\mathcal{E}$ and a regular cardinal κ ,*

- *the full subcategory $\mathcal{C}_\kappa \subset \mathcal{E}$ is essentially small and κ -extensive.*

- the topology $J_{\mathcal{E}}$ on $\mathcal{C}_{\kappa}$ whose covering sieves are jointly epimorphic in $\mathcal{E}$ coincides with the κ -extensive topology $J_{\kappa\text{-ext}}$.

Furthermore, there exists a regular cardinal κ such that $\mathcal{C}_{\kappa}$ is a generating full subcategory of $\mathcal{E}$, if and only if the topos $\mathcal{E}$ has enough projectives.

Proof. First, we prove the essential smallness. Fix a generating set G of the topos $\mathcal{E}$. Lemma 3.9 implies that any object in $\mathcal{C}_{\kappa}$ is a κ -small coproduct of subobjects of objects in G . This implies that there are at most a small number of isomorphism classes in $\mathcal{C}_{\kappa}$.

Lemma 3.1 and Proposition 3.8 imply that $\mathcal{C}_{\kappa} \hookrightarrow \mathcal{E}$ is closed under taking summands and κ -small coproducts. This implies that $\mathcal{C}_{\kappa}$ inherits the κ -extensivity of $\mathcal{E}$.

Lemma 3.9 implies that a sieve $S \subset \mathcal{C}_{\kappa}(-, P)$ belongs to $J_{\mathcal{E}}$ if and only if it contains a κ -small coproduct diagram.

The last statement follows from the fact that every object X in a Grothendieck topos is κ -narrow for a sufficiently large cardinal κ . $\square$

Corollary 3.13 (cf. [MM92, Appendix, Section 4, Corollary 1]). *If a Grothendieck topos $\mathcal{E}$ has enough projectives, we have*

$$\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}_{\kappa}, J_{\kappa\text{-ext}})$$

for sufficiently large regular cardinal κ , where $\mathcal{C}_{\kappa}$ is the essentially small κ -extensive full subcategory consisting of κ -narrow projective objects, and $J_{\kappa\text{-ext}}$ is its κ -extensive topology.

4. CONCLUSION

Theorem 4.1 (Main Theorem). *For a Grothendieck topos $\mathcal{E}$, the following conditions are equivalent:*

- (1) $\mathcal{E}$ has enough projective objects.
- (2) There is a regular cardinal κ such that $\mathcal{E}$ is a κ -Gaeta topos (Definition 1.6).
- (3) $\mathcal{E}$ is equivalent to a sheaf topos $E \simeq \mathbf{Sh}(\mathcal{C}, J)$, where $(\mathcal{C}, J)$ is a small site generated by free sieves.

Proof. Corollary 3.13 proves the implication (1) $\implies$ (2). Proposition 2.5 proves the implication (2) $\implies$ (3). Proposition 2.8 proves the implication (3) $\implies$ (1). $\square$

Example 4.2 (Presheaves). A presheaf topos $\mathbf{PSh}(\mathcal{C})$ has enough projectives, since the trivial topology on $\mathcal{C}$ is generated by free sieves.

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