

TOTALLY DISCONNECTED LOCALES AND TOPOI

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ABSTRACT. This note is about the study of *totally disconnected topoi*, topoi that do not have a non-trivial *connected quotient* (= a *quotient topos*). Totally disconnected topoi turn out to be topoi of sheaves on locales that satisfy a certain kind of total disconnectedness.

This work is in progress and does not constitute the complete version of the project.

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In this paper, a topos always means a Grothendieck topos.

1. PRELIMINARIES: CONNECTED QUOTIENT AS RELATIVE CONNECTEDNESS

Definition 1.1. A geometric morphism $f: \mathcal{E} \rightarrow \mathcal{F}$ is *connected* if f^* is fully faithful. A *connected quotient* is a connected geometric morphism from $\mathcal{E}$.

Geometrically, a connected geometric morphism is a relativized connectedness.

Proposition 1.2 ([AJ21, Section 3.2.14]). *A topos $\mathcal{E}$ is connected, if and only if its global section geometric morphism $!: \mathcal{E} \rightarrow \mathbf{Set} = \mathbf{Sh}(1)$ is a connected geometric morphism.*

2. TOTALLY DISCONNECTED TOPOI AND LOCALES

2.1. Totally disconnected topoi.

Definition 2.1. A topos $\mathcal{E}$ is *totally disconnected* if any connected geometric morphism from $\mathcal{E}$ is an equivalence.

Remark 2.2. We make a remark on preceding research.

- In [AJ21, Section 3.2.15], totally disconnected topoi are called disconnected topoi.

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2.2. Totally disconnected locales. The following definition and proposition allow us to reduce the problem of totally disconnected topoi to the locale theory.

Definition 2.3. A locale X is *totally disconnected* if its sheaf topos $\mathbf{Sh}(X)$ is a totally disconnected topos.

Proposition 2.4. *A topos is totally disconnected, if and only if it is equivalent to the sheaf topos over a totally disconnected locale.*

Proof. Due to the hyperconnected-localic factorization system [Joh81, Theorem 1.4]. $\square$

Lemma 2.5. *The total disconnectedness does not imply being zero-dimensional, being spatial, and total separatedness:*

- *There is a totally disconnected locale that is not zero-dimensional.*
- *There is a totally disconnected locale that is not spatial.*
- *There is a totally disconnected locale induced by a topological space that is not totally separated.*

Proof. Due to the next subsection. $\square$

2.3. Examples of totally disconnected topoi.

Example 2.6. The topos $\mathbf{Set}$ is totally disconnected. This is the only connected and totally disconnected topos.

Example 2.7. For any zero-dimensional topological space¹ X , its sheaf topos $\mathbf{Sh}(X)$ is totally disconnected. For example, $\mathbf{Sh}(\mathbb{Q})$ is totally disconnected [AJ21, Section 3.2.15, Example (vi)]. For any profinite space X , for example, the Cantor space, $\mathbf{Sh}(X)$ is totally disconnected.

Example 2.8. For a totally disconnected topos $\mathcal{E}$, its subtopoi are totally disconnected [AJ21, Section 3.2.15, Example (iv)].

Example 2.9. For a totally disconnected topos $\mathcal{E}$, its étale bundles, i.e., its slice topoi $\mathcal{E}/X$ for an object X in $\mathcal{E}$, are totally disconnected [AJ21, Section 3.2.15, Example (ii)].

Example 2.10. Sheaf topoi over discrete topological spaces are called *discrete topoi*. We can prove that discrete topoi are exactly locally connected and totally disconnected topoi. Let $\mathcal{E}$ be a locally connected and totally disconnected topos. Then we have the decomposition of the terminal object $1_{\mathcal{E}}$ into the coproduct of its connected components $1_{\mathcal{E}} = \coprod_{\lambda \in \Lambda} S_{\lambda}$. Then we obtain $\mathcal{E} \simeq \mathcal{E}/(\coprod S_{\lambda}) \simeq \prod \mathcal{E}/S_{\lambda}$. Since each $\mathcal{E}/S_{\lambda}$ is connected and totally disconnected, we conclude that $\mathcal{E}$ is discrete $\mathcal{E} \simeq \mathbf{Set}^{\Lambda}$.

Example 2.11. Let X be a topological space obtained as a coequalizer of

$$\mathbb{Q} \setminus \{0\} \rightrightarrows \mathbb{Q} \amalg \mathbb{Q} \longrightarrow X.$$

Since X is an étale bundle over $\mathbb{Q}$, its sheaf topos $\mathbf{Sh}(X)$ is totally disconnected. This is an example of totally disconnected topos that is neither zero-dimensional nor totally separated.

Example 2.12. Any *forcing topos*, which is a sheaf topos over the posetal site with the dense (=double negation) topology, is totally disconnected. For example, the Cohen topos is totally disconnected.

Example 2.13. More generally, a localic and boolean topos $\mathcal{E}$ is totally disconnected [Hor24, Proposition 5.7]. For example, for a topological space X , the booleanization of its sheaf topos $\mathbf{Sh}(X)$, which will be denoted by $\mathbf{Sh}(X_{\neg\neg})$, is totally disconnected. In particular, $\mathbf{Sh}(\mathbb{R}_{\neg\neg})$ is a non-trivial example of a totally disconnected topos without points. This topos exemplifies that a totally disconnected topos might not be a sheaf topos over a topological space.

¹A topological space is *zero-dimensional*, if any open subset is a union of clopen subsets.

Example 2.14. A topological space X is *totally path-disconnected*, if any path $[0, 1] \rightarrow X$ is constant. Obviously, any totally disconnected space is totally path-disconnected. However, the converse is not true:

Take an infinite set A whose cardinality is strictly larger than the continuum cardinality $|A| > 2^{\aleph_0}$. Considering the $\text{co-}2^{\aleph_0}$ topology, i.e., a subset $S \subset A$ is closed if $|S| \leq 2^{\aleph_0}$ or $S = A$, we regard A as a topological space. This space A is connected.

We prove the topological space A is totally path-disconnected. Take an arbitrary path $p: [0, 1] \rightarrow A$. If its image $p([0, 1])$ is not a singleton, we can divide it into a non-trivial disjoint union $p([0, 1]) = S \amalg S'$, where $S, S' \neq \emptyset$. By the definition of the topology of A , two subsets $S, S' \subset A$ are closed, hence their inverse images $p^{-1}(S), p^{-1}(S') \subset [0, 1]$ are closed. Since $p^{-1}(S), p^{-1}(S') \subset [0, 1]$ are non-empty, closed, and complements of each other, this contradicts the connectedness of the unit interval $[0, 1]$.

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